

Modeling Multivariate Time Series in Economics: from Auto-Regressions to Recurrent Neural Networks*

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Abstract

The modeling of multivariate time series in an agnostic manner, without assumptions about underlying theoretical structure is traditionally conducted using Vector Auto-Regressions. They are well suited for linear and state-independent evolution. A more general methodology of Multivariate Recurrent Neural Networks allows to capture non-linear and state-dependent dynamics. This paper takes a range of small- to large-scale Long Short-Term Memory MRNNs and pits them against VARs in an application to US data on GDP growth, inflation, commodity prices, Fed Funds rate and bank reserves. Even in a small-sample regime, MRNN outperforms VAR in forecasting: its out-of-sample predictions are about 20% more accurate. MRNN also fares better in interpretability by means of impulse response functions: for instance, a shock to the Fed Funds rate variable generates system dynamics that are more plausible according to conventional economic theory. Additionally, the paper shows how, due to its inherent non-linearity, MRNN can discover (in an unsupervised manner) different macroeconomic regimes. Utilizing its state dependence, MRNN may also be a useful tool for policy simulations under practically relevant economic conditions (such as Zero Lower Bound).

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1 Introduction

Artificial Neural Networks are mathematical models for pattern extraction and recognition, a task they achieve by passing their inputs through a series of non-linear transformations and mapping them to targeted outputs; hence the term Deep Learning. Over the last several years this machine learning approach demonstrated impressive progress in image, text/sound and video recognition. Success in image processing became possible due to Convolutional Neural Networks that work well for spatial data. Success in text and sound processing was made to a large extent due to Recurrent Neural Networks that work well for sequential data. This paper investigates ANNs applicability to multidimensional time series data that are encountered in economics, naturally focusing on the RNN family, and on its Long Short-Term Memory member more specifically.

Recurrent Neural Network maintains within its network the connections that look back in time. When processing sequential data, this allows RNN to account for a relatively distant history, to keep track of the context, and to handle data of multiple frequency or scales. For example, in a natural language processing application within the context of geography-related words such as “city” or “region” an RNN after encountering the word “French” would predict a high probability of encountering word “Toulouse” next, but within the context of food-related words such as “bread” or “butter” it would predict instead a high probability of “baguette” coming next.

In this paper we are interested in agnostic statistical modeling that does not rely on “structural” restrictions stemming from economic theory. Traditionally in economics this is implemented using Vector Auto-Regressions, a simple time-series model that we use as a benchmark. On the other hand, LSTM Multivariate Recurrent Neural Network fulfills the same role, additionally allowing for (i) highly non-linear relationships, as well as (ii) state-dependent dynamics in the data. In fact, a VAR model can be viewed as a special case of MRNN.

The data set we use here comprises GDP growth, inflation, commodity prices, Fed Funds rate, and bank reserves for the United States since 1959 (at monthly frequency, after the aggregation/disaggregation procedures described in the text).

In our exercise MRNN architectures of varying depth (from 1 to 4 hidden layers) and width (from 5 to 500 units per layer) are considered. In the case of large-capacity NNs, we allow for substantial redundancy, since available regularization techniques such as dropout prevent overfitting quite effectively. MRNN is trained using mini-batch gradient descent. VAR is estimated by OLS, trying various lag lengths. In both cases, the process of parameter learning is guided by prediction performance on the validation sub-sample.

Basically, we run a genuine horse-race between VAR and MRNN, providing two models with the same data sets and letting them use the data in the most effective way their structure allows. Then we compare the fruitfulness of the two modeling approaches.

Our preferred LSTM MRNN architecture is fairly minimalistic, comprising 2 hidden layers with 10 units in each. VAR specification we ended up with comprises 4 lags.

This paper’s approach is rooted in modern machine learning that focuses on predicting data out-of-sample, as opposed to more traditional econometrics that deals with fitting data in-sample. Because of this, and also for the reasons of practical interest, the first, quantitative domain for comparing two modeling methods is prediction performance. Our LSTM MRNN outperforms VAR by about 20% in prediction accuracy on the testing sub-sample (i.e., in generalization).

The second, qualitative domain is related to an aspect of model interpretation that is revealed by impulse responses. These are a standard exercise for VARs, and in this paper we propose a simple procedure of their implementation for MRNNs. We examine the commonly considered situation of an increase (transitory or permanent) in the Fed Funds rate, and verify which of the two models behaves more in line with economic intuition.

It turns out that following such a monetary tightening, the reactions produced by MRNN are arguably more plausible than those produced by VAR. For the output growth, VAR implies an initial acceleration, which is followed by deceleration later. MRNN implies deceleration unambiguously. As far as inflation rate is concerned, VAR implies its unambiguous acceleration. MRNN is more nuanced, it implies an inflation slowdown in the state of the world characterized by low inflation and low interest rates as well as high growth rates, but a mixed reaction in the state of the world characterized by high inflation and high interest rates as well as low growth rates.

Putting this into context, a pick-up in output growth in the wake of monetary contraction is hard to justify theoretically. A positive response of prices to monetary tightening is also a theoretically controversial finding, which has been referred to as a “price puzzle” in VAR literature (Sims, 1992; Uhlig, 2005), although some authors do offer variations of Fisherian arguments to justify it (Cochrane, 2018). Thus, in order to exhibit response dynamics that are more plausible, VARs usually require an introduction of additional theoretical (“structural”) identification restrictions. However, MRNN appears to identify plausible reactions essentially from raw data, without the need to impose any theoretical structure beyond a simple recursive ordering of variables.

Next, we move beyond the comparison of the two modeling approaches, and touch upon MRNN applications and interpretations. Firstly, a primer on policy simulation. Using the MRNN machinery for modeling conditional dynamics, we can examine the effect of monetary tightening in the special circumstances when the Fed Funds rate is close to its Zero Lower Bound. In such case, the permanent monetary tightening shock has a less negative effect on output growth and inflation rate than in “normal” times.

Secondly, some additional tools for analysis and interpretation. Apparently, MRNN has discovered in the data two different macroeconomic regimes, one characterized by low inflation and interest rates, and another characterized by high inflation and interest rates. Interestingly, it has done so under a completely “hands-off” procedure, without any special modeler’s intervention (such as explicitly specifying a regime-switching model).

Lastly, it is worth noting that we have a fairly small data set: just 59 years of monthly-

frequency data on 5 economic variables. The parameterizations of our two models are quite rich: VAR comprises 105, while MRNN comprises 1535 parameters. The conventional views are skeptical about the viability of statistical models in such situations, especially with regard to “data-hungry” NN architectures. However, there are theoretical arguments why NNs effective capacity is in practice substantially lower than what a raw parameter count suggests (see the main body); this point is indirectly supported by strong performance of our MRNN in out-of-sample predictions. In any case, we did attempt a data augmentation exercise, as one way to synthetically expand the limited data set; it did not seem to bring any benefit though.

2 Literature

Artificial Neural Networks and Deep Learning is a very rapidly advancing area of knowledge. Excellent textbook treatment is available in a still relevant Bishop (2006), as well as in a less formal but more up-to-date Goodfellow et al. (2016). A very broad review is available in Schmidhuber (2015), a good review with an eye on applications to Natural Language Processing is Goldberg (2016), a review of Recurrent Neural Networks is offered in Lipton et al. (2015).

Important ANN architectures include Feed Forward Neural Networks that were historically first (single-layer due to Rosenblatt, 1958; multi-layer proposed by Ivakhnenko and Lapa, 1965), Convolutional Neural Networks used for image recognition (LeNet introduced by LeCun et al., 1998; AlexNet by Krizhevsky et al., 2012; GoogLeNet by Szegedy et al., 2014), Recurrent Neural Networks used for text/sound processing (Elman, 1990). Here we focus on the latter. (Interestingly, according to Liao and Poggio (2016), RNNs are biologically-plausible models of the cortex—thus closing the loop back to seminal McCulloch and Pitts (1943), whose pioneering NN ideas lied on the interface of biology and computer science.)

Recurrent Neural Network forms a network that loops to itself, which allows to account for recent history when processing sequential data. Specifically, we focus on Long Short-Term Memory variation of RNN, which has the facilities to prevent the problem of vanishing/exploding error gradients used in the Back-Propagation Through Time algorithm (Hochreiter and Schmidhuber, 1997).

Apart from theoretical existence results regarding accurate approximation (Cybenko, 1989; Hornik et al., 1989; also see Kolmogorov, 1957; and Arnold, 1957), there is still no consensus about the reasons for NN effectiveness. Possible explanations come from statistical physics (Mehta and Schwab, 2014), geometry (Lei et al., 2018), information theory (Schwartz-Ziv and Tishby, 2017), Bayesian statistics (Patel et al., 2015; Polson and Sokolov, 2017; Friston et al., 2018). Arguably, an important role is played by clever training and regularization techniques (such as Hinton and Camp, 1993; Hochreiter and Schmidhuber, 1997; Srivastava et al., 2014; Zaremba et al., 2015; Mandt et al., 2017; and

many others).

Besides well-known practical applications, ANNs gain wide usage in natural science and mathematics. For instance, RNNs have been successfully applied to high-dimensional chaotic systems (Jaeger and Haas, 2004; Vlachas et al., 2018; Pathak et al., 2018). NNs also found application in cryptography (Maghrebi et al., 2016; Cagli et al., 2017), and even to data from geometry and physics that is relevant to string theory (He, 2017).

Economics is also not lagging behind. ANNs have been used for estimation of causal relationships developing the broad idea of Instrumental Variables (Hartford et al., 2016), for auction design (Dütting et al., 2017), for finance (Heaton et al., 2016a; 2016b; Sirignano, 2016; Feng et al., 2018; Gu et al., 2018).

Considering applications to sequential data, a good source is Graves and Schmidhuber (2009), who use multidimensional LSTM RNN for handwriting recognition. A more recent paper is Goel et al. (2016), with their application of LSTM to multivariate time series from aviation industry. One application to sequential data from the field of finance is Dixon et al. (2018), although instead of RNN the authors use a simpler FFNN with lagged explanatory variables embedded into an input vector.

As a benchmark for comparison, we rely on Vector Auto-Regressions. Loosely speaking, they combine Auto-Regressive Integrated Moving Average models (Box and Jenkins, 1970) and models comprising systems of equations (Theil, 1954; Zellner, 1962; Zellner and Theil, 1962). In economics, this approach was popularized by Sims (1980). Thanks to their simplicity, VARs are still extensively used. A recent review of VAR and time series analysis more generally is available in Stock and Watson (2017), detailed reviews of the so-called structural VARs are offered by Ramey (2016) as well as Stock and Watson (2016).

In this paper we apply MRNN and VAR models to macroeconomic data from the United States, with an eye on forecasting and policy analysis. Relevant references are provided in the text sections where these applications are discussed in detail.

3 Methodology

3.1 Framework

First, let us clarify some of the terminology used going forward. We divide the whole universe of Artificial Neural Network models into different architectures (or specifications), and we divide each architecture into its different trained instances.

Given an architecture, the process of hyperparameter tuning (a kind of meta-optimization) defines the parameters ruling the optimization process.¹ Then, optimization, or learning, algorithm executes the model's parameter training (as opposed to estimation

¹Although sometimes hyperparameters are understood as also including the details of the architecture itself.

in more traditional statistical settings). Next, regularization techniques help with the trained model’s parameter refinement (note that optimization and regularization are often conducted in an alternating manner). Lastly, model selection means: (a) for a specific architecture, the choice among trained instances of the best one; and/or (b) between different architectures, the selection of the best one.

3.1.1 Architecture

Artificial Neural Networks, or just nets, are inspired by biological neural networks: units (nodes, cells or “neurons”) have connections (edges or “axons”, “synapses” and “dendrites”) to other units, and they perform complex non-linear computations by transmitting signals from one to another. Mathematically, ANN is a model defined by the following system of equations (or architecture):

$$\begin{aligned}
\mathbf{h}_1 &= \mathbf{g}_1(\mathbf{b}_1 + \mathbf{W}_1 \mathbf{x}), \\
\mathbf{h}_2 &= \mathbf{g}_2(\mathbf{b}_2 + \mathbf{W}_2 \mathbf{h}_1), \\
&\dots \\
\mathbf{h}_M &= \mathbf{g}_M(\mathbf{b}_M + \mathbf{W}_M \mathbf{h}_{M-1}), \\
\hat{\mathbf{y}} &= \mathbf{g}_{M+1}(\mathbf{b}_{M+1} + \mathbf{W}_{M+1} \mathbf{h}_M),
\end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^{K_x}$ is a K_x -dimensional input vector; $\mathbf{b}_m \in \mathbb{R}^{n_m}$ and $\mathbf{W}_m \in \mathbb{R}^{n_m \times n_{m-1}}$ are parameters (biases and weights, respectively); $\mathbf{g}_m(\cdot)$ are activation functions such that $\mathbf{g}_m : \mathbb{R}^{n_m} \mapsto \mathbb{R}^{n_m}$; $\mathbf{h}_m(\cdot)$ are outputs of hidden layers; with hidden layers $m \in \{1, \dots, M\}$ and hidden units $n_m \in \mathbb{N}^+$ (keeping $n_0 := K_x$); as well as $\hat{\mathbf{y}} \in \mathbb{R}^{K_y}$ is the output vector aiming to predict the value of a random variable $\mathbf{y} \in \mathbb{R}^{K_y}$, with $\boldsymbol{\xi}$ being the prediction error,

$$\mathbf{y} := \hat{\mathbf{y}} + \boldsymbol{\xi}. \tag{2}$$

In this paper, $K_x = K_y =: K \in \mathbb{N}^+$.

The above system can be more compactly expressed as a composition of functions

$$\hat{\mathbf{y}} = g_M^B \circ \dots \circ g_1^B(\mathbf{z}), \tag{3}$$

where, for $\mathbf{B}_m := [\mathbf{b}_m, \mathbf{W}_m]$ and $\mathbf{z} := [1; \mathbf{x}]$, we used the notation

$$\mathbf{g}_m^B(\mathbf{z}) := \mathbf{g}_m(\mathbf{B}_m \mathbf{z}). \tag{4}$$

Basically, the network passes the incoming data \mathbf{x} through several stages of non-linear (semi-linear) transformations, and aims to predict some target \mathbf{y} . One can think of $\mathbf{g}_m(\cdot)$ as units, and of \mathbf{W}_m as connections.

For the time-series setup, we are using a particular type of ANN that is known as (Multivariate) Recurrent Neural Network, in particular the specification called Long Short-

Term Memory.² It specializes the general model along the following lines. First, random variables as well as state variables are time-indexed. Second, the input variable is just a lagged variable that is being predicted by the output, $\mathbf{x}_t := \mathbf{y}_{t-1}$. Third, $\mathbf{g}_{m,t}(\cdot)$ takes a special form (for readability, omitting the layers' subscripts m throughout):

$$\begin{aligned}
\mathbf{h}_t &= \mathbf{o}_t \odot \tanh(\mathbf{c}_t), \\
\mathbf{c}_t &= \mathbf{f}_t \odot \mathbf{c}_{t-1} + \mathbf{i}_t \odot \tanh(\mathbf{b}_c + \mathbf{W}_c \mathbf{h}_{-1,t} + \mathbf{U}_c \mathbf{h}_{t-1}), \\
\mathbf{o}_t &= \sigma(\mathbf{b}_o + \mathbf{W}_o \mathbf{h}_{-1,t} + \mathbf{U}_o \mathbf{h}_{t-1}), \\
\mathbf{i}_t &= \sigma(\mathbf{b}_i + \mathbf{W}_i \mathbf{h}_{-1,t} + \mathbf{U}_i \mathbf{h}_{t-1}), \\
\mathbf{f}_t &= \sigma(\mathbf{b}_f + \mathbf{W}_f \mathbf{h}_{-1,t} + \mathbf{U}_f \mathbf{h}_{t-1});
\end{aligned} \tag{5}$$

where $\mathbf{h}_t \in \mathbb{R}^n$ is in this specification interpreted as a cell hidden state vector (more precisely, due to the readability caveat above, \mathbf{h}_t stands for $\mathbf{h}_{m,t}$, and $\mathbf{h}_{-1,t}$ denotes $\mathbf{h}_{m-1,t}$, while $\mathbf{h}_{0,t} := \mathbf{x}_t$); $\mathbf{c}_t \in \mathbb{R}^n$ is a cell memory state vector; $\mathbf{i}_t \in \mathbb{R}^n$, $\mathbf{f}_t \in \mathbb{R}^n$ and $\mathbf{o}_t \in \mathbb{R}^n$ are an input gate, forget gate and output gate activation vectors, respectively; and with $\mathbf{b}_\cdot \in \mathbb{R}^n$, $\mathbf{W}_\cdot \in \mathbb{R}^{n \times n-1}$, and $\mathbf{U}_\cdot \in \mathbb{R}^{n \times n}$ being parameters.

Introduction of state vectors allows the time-series models to condition on recent historical dynamics. It may be helpful to think of LSTM network as a kind of non-linear state-space model (that are very easy to work with due to Kalman-Bucy filter as described, for example, in Hamilton, 1994).

3.1.2 Mechanics

An important theoretical result about NNs is that their formulation is general enough to represent any desired function. Given at least one hidden layer and a sufficiently large but finite number of units, this is feasible by the arguments of the universal approximation theorem (Cybenko, 1989; Hornik et al., 1989; also see the representation theorem due to Kolmogorov, 1957; and Arnold, 1957).³ A useful implication is NNs' ability to account for non-linearity of the empirical phenomena being modeled.

Size (or capacity, parameterization) of a NN is determined by its depth, that is the number of hidden layers M , and its width, that is the number of units in each layer $\{n_1, \dots, n_M\}$. NNs are called deep if they have a large number of hidden layers.

The above theory requires a large enough NN, but it does not provide us with the formal method for setting a sufficient size and structuring the architecture (and, more generally, for learning the required parameters).⁴ Our guiding principle is given by what is regarded as the scientific method: successful prediction on the new data. A theory or

²In the specification's title, "long memory" stands for slowly-changing parameters of the ANN, "short memory" for recurrent inputs to the hidden layer from its past outputs, and "long short-term memory" for the amendment of recurrent connections with additional parametric network structures.

³Formally, a function that is Borel measurable and defined over finite-dimensional spaces (e.g., continuous on a compact subset of \mathbb{R}^n) can thus be approximated with arbitrary precision.

⁴Technically, the above theoretical result is not constructive, it is only about the existence.

model is good only if it generalizes the existing facts in a way that is useful in predicting yet unobserved phenomena.

From the theory, we know that an insufficiently sized NN will underfit the data: the standard statistical fit measures will demonstrate poor performance, even on the observed sample that is used for training (“in-sample”). At the other extreme, a too large NN may overfit the data: statistical fit measures will demonstrate strong performance on the observed sample that was used for training, but will collapse when the model is tested on the previously unobserved data (“out-of-sample”). In the machine learning literature, we say about the latter case that the NN has memorized the data without learning its high-level structure and thus acquiring the ability to generalize the accumulated knowledge.

Thus, the strategy commonly adopted in the literature is extensive exploration that amounts to experimenting with different model specifications, using as a criterion that guides the exploration the predictive performance of the specifications considered. Usually, the predictive fit is measured on a sample reserved exclusively for such validation purposes (and that is separate from the training and testing samples).

One practical approach is to start with a generously sized network, allowing for redundancy and erring on the side of a too many parameters, and then restricting the excessive depth/width using the regularization methods described below.⁵ Another practical rule of thumb is that deep architectures tend to be easier and cheaper to train, hence their popularity even though in theory having just one hidden layer is sufficient.

Regularization: The aim of regularization is to improve generalization and reduce overfitting of a given model. The leading motive is Occam’s razor principle: the heuristic postulate that good scientific theories are parsimonious. Regularization downplays the weight of the training sample in parameter learning by introducing additional information and constraints, which tends to prevent the fitting to noise and memorization at the expense of generalization.

The recent literature provides many effective regularization techniques. Some of the most popular methods are explicitly using the idea that good models should be invariant to “small” perturbations.

Dropout adds noise to the NN architecture (Srivastava et al., 2014; also see Zaremba et al., 2015). During the training process, it randomly drops from the model some units and connections (usually in the hidden layer, but possibly in the visible one too). This is a very simple yet highly effective regularization method. It can be loosely motivated as mimicking the model averaging (as the authors of Srivastava et al., 2014, originally suggested) as well as estimation under LASSO/sparsity or Ridge constraint (Wager et al.,

⁵Existence of effective regularization methods is the reason why overfitting is less problematic than underfitting (this is a little reminiscent of inefficiency versus inconsistency in the practice of estimating linear regression models, hence the prudent preference for including rather too many than too few explanatory variables). Also, allowing for redundancy improves NN reliability and makes it easier to train.

2013).

Data augmentation adds noise to the NN inputs (the idea can be traced back to Sietsma and Dow, 1991).⁶ This expands the data set with synthetic data comprising the originals that are contaminated with some (usually, Gaussian/white) noise. In such case, effectively we are dealing with a kind of background noise. But noise can also be added exploiting the structure of the data at hand. For visual data, it involves image transformations such as translations, rotations, scaling, cropping, color perturbations (LeCun et al., 1998; Simard et al., 2003; Krizhevsky et al., 2012). For audio data, it involves sound transformations such as frequency, tempo and speed perturbations (Jaitly and Hinton, 2013; Cui et al., 2015; Ko et al., 2015).

Optimization and tuning: NN models are very costly to work with: usually they are truly high-dimensional and have a large number of parameters; they have non-linear structure and non-convex objective functions; they often involve processing very large datasets. Thus, careful design of the whole computational approach is of paramount importance. This paper leaves out the discussion of the hardware and software components (necessary resources such as Graphics Processing Units and machine learning libraries are now widely available), and focuses instead on the algorithmic component that comprises data handling, minimization algorithms, meta-optimization of the latter, etc.

We briefly present some of the techniques that proved to be useful in training NN models. First and foremost, their aim is to speed up training and reduce computational costs.⁷ However, many of the below techniques involve replacing a true algorithm or object with some reasonable approximation, and these techniques' effectiveness is possible only if they (or the respective downstream algorithms) are invariant to perturbations and robust to random noise that necessarily show up. In other words, along the way these speed-up techniques also act as regularizers.

The loss minimization algorithms in the existing literature are usually variations of a standard gradient descent with some stochasticity added (for an early source, see Robbins and Monro, 1951; for a Bayesian interpretation see Mandt et al., 2017).

The granularity of data inputs used for gradient computations and parameter updates vary from utilizing the full data set ("batch gradient descent") to several ("mini-batch gradient descent") or single ("stochastic gradient descent") randomly chosen block(s) of observations (that is, training example(s)). Smaller batches prevent using the most efficient computational tools and exhibit noisier convergence; but such built-in stochasticity helps avoiding local minima, hence works as a regularizer (see Wilson and Martinez, 2003).

Data standardization is useful in many statistical learning problems, but in the context

⁶Data augmentation is often viewed intuitively as an addition of carefully perturbed synthetic data to the existing data set. Formally, however, it is a regularization method that encourages fitting to relevant features of the data rather than to irrelevant noise.

⁷Hence, such techniques are often just practical tricks that are employed only out of necessity and computational resource constraints.

of NNs it is even more critical. One development of the general standardization idea that is tailored for NNs is separately standardizing the inputs to intermediate hidden layers (“batch normalization”, see Ioffe and Szegedy, 2015).

Minimization algorithms make use of efficient ways for computing gradients that improve upon the canonical chain rule (“backward propagation of errors”, or just “back-propagation”, see Rumelhart et al., 1986).

There are also methods used for ensuring minimization algorithms’ satisfactory convergence, i.e. one that is fast and avoids bad local minima. These include clever ways of updating parameters in the face of stochasticity inherent to minimization algorithms used: the learning rate is high far from the minimum in order to ensure fast learning, but falls as the minimum is approached to prevent overshooting (“learning rate decay”, see Robbins and Monro, 1951; Welling and Teh, 2011); parameter updating maintains memory of recent updates in order to keep improving in the promising direction (“momentum”, see Bengio et al., 2012; Sutskever et al., 2013). Also, simple as it is, an outright termination of the algorithm’s iterations well before the minimum is reached saves on computational costs with little downside (“early stopping”, see Prechelt, 1997). Importantly, all of the above methods contribute to regularization as well.

Many of the methods above have their own hyperparameters (random seeds, dropout probability, learning rates, etc.). The hyperparameters’ initial values—and sometimes modification schedules too—are also important choices to be made.

“Overparameterization”: A common question about NNs (e.g., see Zhang et al., 2017; Belkin et al., 2018) is concerned with the fact that in overwhelming majority of applications the number of NN parameters exceeds the number of training observations by several orders of magnitude — what are the underlying mechanisms that allow NNs with large explanatory capacity that fit arbitrarily accurately on the training sample also to perform well in the test sample, i.e. to properly generalize?

Formally, the basic explanation is that all these parameters are not entirely free. For instance, there are some trivial symmetries and equivalences in the space of NN weights (Chen et al., 1993; Kurková and Kainen, 1994). More importantly, parameters of properly trained NNs turn out to lie in a lower-dimensional subspace (due to “noise stability” property of Arora et al., 2018; also see Hochreiter and Schmidhuber, 1997, Hinton and Van Camp, 1993, for a related argument).

3.2 Data

Our raw data set comprises: Real GDP, seasonally adjusted annual rate, and GDP Price Deflator, seasonally adjusted (both at quarterly frequency, sourced from Federal Reserve)⁸; Equal Weight Continuous Commodity Index (at daily frequency, sourced

⁸Seasonally adjusted data are used for simplicity, because ANNs’ treatment of seasonality in a small-sample regime as here is an important research problem in its own right and deserves a separate study.

from Thomson Reuters); Effective Fed Funds Rate and Total Nonborrowed Reserves (at monthly frequency, sourced from Federal Reserve). Daily frequency data are aggregated into monthly by averaging, while quarterly frequency data are disaggregated into monthly using a linear state-space model (as in Bernanke et al., 1997). Next, data are transformed as follows: Real GDP, GDP Deflator and Commodity Price Index into month-over-month growth rates; Nonborrowed Reserves into monthly differences. (Additionally, to prevent numerical complications, the training itself is done in terms of data series that are standardized to have a mean of 0 and a variance of 1.)

See Figures 6 and 7 in Appendix §A for visual representations of monthly-frequency data prior and after the transformations, respectively.⁹

The resulting data set covers the time period from 1959:M2 to 2018:M03.

3.3 Implementation

We split those 59 years of monthly data into 3 sub-samples:

- (i) training sample used for parameter learning (1959:M2 to 2013:M3);
- (ii) validation sample used for adjusting the learning rate in order to prevent data memorization (2013:M4 to 2015:M9);
- (iii) testing sample that the NN does not see until we are finished with learning and start benchmarking NN’s usefulness (2015:M10 to 2018:M03).

Here we have sequential data, i.e. each observation is dependent on other observations immediately prior to it. Hence, one training example includes several (specifically, 24) consecutive observations: $\{\mathbf{y}_{t-\ell}\}_{\ell=0}^{23}$.

Because of data dependence, predictions over the validation and testing sub-samples are made multiple steps ahead (splitting each sub-sample in half and predicting 15 steps ahead, thus reserving 2×15 months for validation and 2×15 months for testing).

We use the “golden-standard” mini-batch gradient descent, with 4 training examples per batch. Then, 1 training epoch is composed of a full run through the training data set, thus feeding the training examples for all possible t : $\{\{\mathbf{y}_{t-\ell}\}_{\ell=0}^{23}\}_{t=24}^{650}$.

Our NN has a general LSTM architecture introduced earlier. More specifically, NN cells in the visible input layer take in the 5-dimensional data vector (so, $K := 5$) and pass it on untransformed. The cells in the hidden layers use Rectified Linear Units (ReLU) as the activation functions applied to their inputs. The cells in the visible output layer apply simple linear transformations as activation functions on their inputs from the last hidden layer; with the Mean Squared Error loss function ultimately used in our NN, in this final visible layer we are effectively just running Ordinary Least Squares regressions on the layer’s inputs.

⁹Notice the effect of quantitative easing policy on Nonborrowed Reserves at the end of the sample period.

Within certain range, we try architectures of varying depth and width. We consider uniform architectures, as well as the ones with squeezed (“bottleneck” layer used for dimension reduction; see Rumelhart et al., 1985, and Boulard and Kemp, 1988; also see usage examples in Hinton and Salakhutdinov, 2006, and Szegedy et al., 2014) and expanding shapes (“sparse” representation; see Ranzato et al., 2007; also see usage example in Delahunt and Kutz, 2018).

The minimization algorithm used is adaptive moment estimation, or Adam (Kingma and Ba, 2014). Each architecture is trained for 20 epochs, with a decaying learning rate. Lastly, the dropout probability is set at 0.2.¹⁰

To sum up, our implementation recipe has four stages: (i) set hyperparameters (which requires some tuning); (ii) define specification, train and regularize it; (iii) repeat the previous step for a range of specifications; (iv) choose specification with best training fit.¹¹

3.4 Benchmark

For comparison, we use a standard Vector Auto-Regressive model. VAR model is defined by the following vector-valued equation:

$$\hat{\mathbf{y}}_t = \mathbf{a}_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \dots + \mathbf{A}_L \mathbf{y}_{t-L}, \quad (6)$$

where $\mathbf{y}_{t-\ell}$ are lags of a K -dimensional random variable \mathbf{y}_t ; while $\mathbf{a}_0 \in \mathbb{R}^K$ and $\mathbf{A}_\ell \in \mathbb{R}^{K \times K}$ are parameters; with lag $\ell \in \{1, \dots, L\}$; and $\hat{\mathbf{y}}_t$ on the left-hand side has the same interpretation as before in equation (2).

More compactly, for $\mathbf{B} := [\mathbf{a}_0, \mathbf{A}_1, \dots, \mathbf{A}_L]$ and $\mathbf{z}_t := [1; \mathbf{y}_{t-1}; \dots; \mathbf{y}_{t-L}]$, and following the notation from (4), this becomes

$$\hat{\mathbf{y}}_t = \mathbf{B} \mathbf{z}_t = \text{id}^B(\mathbf{z}_t), \quad (7)$$

with the identity function defined as

$$\text{id}(\mathbf{z}) := \mathbf{z}.$$

From this perspective, it becomes clear that VAR can be seen as a restricted version of a more general multivariate NN model (with MRNN and LSTM MRNN adding further enrichments).

We apply this model to the data set described earlier. Thus, $K = 5$ again. We split the sample the same way as before into training (1959:M2 to 2013:M3), validation (2013:M4 to 2015:M9) and testing (2015:M10 to 2018:M03) sub-samples. Parameters are estimated by Ordinary Least Squares.

¹⁰Additional hyperparameter settings are available upon request.

¹¹We do not use as a criterion for choosing the best NN specification the validation sample fit (because our validation set is too small for that) or the best training epoch (because learning rate decay makes this extraneous).

Importantly, we follow the “predictive” approach from above and choose the optimal lag length L by comparing the predictive performance on the validation set. Such an explicitly predictive approach differs from the more traditional lag selection practice in VAR literature that is based on various information criteria: the latter also hopes to achieve the generalization ability, albeit indirectly and implicitly. (In the interest of transparency, below we also provide the results for the traditional approach: it selects the lag length using Akaike Information Criterion, combining the training and validation sub-samples together into a larger estimation sample.)

4 Results

4.1 All nets

After tuning the hyperparameters, we trained and regularized a range of architectures, from a modest NN with just 1 hidden layer of 5 units to a fairly rich one comprising 4 hidden layers with 500 units in each. Table 1 summarizes their fitting and prediction performance, measured in terms of Mean Squared Errors for standardized series, over the training, validation and testing samples.

Evidently, a lot of architectures achieve a fairly similar performance. (Also, training results are sensitive to the random seed used, due to the nature of the problem as well as that of the optimization techniques.) Such a situation usually calls for ensemble averaging approaches. However, given that we are already using dropout regularization, which effectively conducts a kind of model averaging, we will abstain from additional mixing.

Moreover, notice that specifications with larger capacities not only exhibit good training sample fit, but also do not necessarily underperform in their testing sample predictions. This corroborates the effectiveness of our regularization techniques.

4.2 Best net

4.2.1 Forecasting

We take the trained NN instance with the lowest training sample MSE as our preferred NN architecture.

With the MSE of 0.6311, this turns out to be a fairly minimalistic specification featuring 2 layers, each including just 10 units. Under the hood, such a specification comprises 1535 NN parameters in total (refer to equations 1 and 5):

- $(10+50+100) + (10+50+100) + (10+50+100) + (10+50+100) = 640$ parameters in the first hidden layer;
- $(10 + 100 + 100) + (10 + 100 + 100) + (10 + 100 + 100) + (10 + 100 + 100) = 840$ parameters in the second hidden layer; and

Table 1: Long Short-Term Memory Multivariate Recurrent Neural Nets, Prediction Results

I				II				III				IV			
n_1	Train.	Valid.	Test.	n_1-n_2	Train.	Valid.	Test.	$n_1-n_2-n_3$	Train.	Valid.	Test.	$n_1-n_2-n_3-n_4$	Train.	Valid.	Test.
5	0.7448	5.0545	4.1719	5-5	0.7151	5.6903	4.1039	5-5-5	0.7544	5.1885	4.0098	5-5-5-5	1.1079	5.8338	4.4154
								5-10-5	0.7827	5.2218	4.0399	5-10-10-5	0.7800	5.3570	4.1138
								10-5-10	0.7173	5.1455	3.8792	10-5-5-10	0.8601	5.8477	4.5416
10	0.6880	5.2903	3.9699	10-10	0.6311	5.1148	4.0858	10-10-10	0.6848	5.1855	4.0427	10-10-10-10	0.8211	5.5939	4.2782
								10-25-10	0.6865	5.3224	5.8654	10-25-25-10	0.7850	5.4949	3.9825
								25-10-25	0.6805	4.9532	4.0537	25-10-10-25	0.8244	5.0129	3.9364
25	0.6818	5.1748	3.9815	25-25	0.6431	5.6948	3.9727	25-25-25	0.6885	5.0445	4.3594	25-25-25-25	0.7851	5.1068	4.0044
								25-50-25	0.6596	5.0147	4.2529	25-50-50-25	0.6842	5.0778	4.2164
								50-25-50	0.6624	5.4168	4.9084	50-25-25-50	0.7395	4.8835	4.4527
50	0.6530	5.1150	3.8830	50-50	0.7071	5.2178	4.4503	50-50-50	0.6679	5.1518	4.0167	50-50-50-50	0.9126	5.9132	4.6092
								50-100-50	0.6462	5.1015	4.1877	50-100-100-50	0.6510	5.3037	4.0904
								100-50-100	0.7986	5.8369	4.5926	100-50-50-100	0.8387	5.2875	4.0926
100	0.6913	5.0546	4.7636	100-100	0.7245	5.1625	4.2298	100-100-100	0.6889	5.5366	4.7786	100-100-100-100	0.6723	5.1201	4.1073
								100-200-100	0.8184	5.3113	4.3729	100-200-200-100	0.7485	8.4162	4.4783
								200-100-200	0.7569	5.3580	4.4943	200-100-100-200	0.8014	5.3540	4.1195
200	0.7320	5.0373	4.1997	200-200	0.7631	4.9071	4.6773	200-200-200	0.7294	6.3925	7.0641	200-200-200-200	0.6501	5.3566	4.6297
								200-300-200	0.7934	4.9213	4.2428	200-300-300-200	0.6947	5.2851	4.1295
								300-200-300	0.8057	5.7319	5.4270	300-200-200-300	1.1478	7.9611	6.8660
300	0.7812	4.9054	3.8704	300-300	0.7920	4.9043	4.0495	300-300-300	0.7999	6.0067	4.3594	300-300-300-300	0.7528	7.6135	9.8760
								300-400-300	0.8138	5.2597	4.4740	300-400-400-300	0.6722	5.6456	5.0593
								400-300-400	0.8197	5.1196	4.8146	400-300-300-400	0.7133	15.5906	10.3539
400	0.7879	5.1756	3.9187	400-400	0.7916	4.9197	4.1388	400-400-400	0.9105	5.4103	7.7445	400-400-400-400	0.6499	4.7265	4.1539
								400-500-400	0.7620	5.1613	4.7135	400-500-500-400	0.7153	7.8092	8.9807
								500-400-500	0.8272	4.8364	5.0482	500-400-400-500	0.6600	5.0797	4.5522
500	0.8375	5.0924	3.9175	500-500	0.8012	4.7628	4.1674	500-500-500	0.8078	5.2696	4.7725	500-500-500-500	0.8630	12.6689	5.8035

Notes: Panels I to IV present the results for NNs of different depth, from 1 to 4 hidden layers, respectively. The rows present the results for NNs with different numbers and permutations of units, from $n_1 = 5$ in one hidden layer to $[n_1, n_2, n_3, n_4] = [500, 500, 500, 500]$ in, correspondingly, hidden layers 1, 2, 3, 4. For each architecture, training ran for 20 epochs. MSEs (calculated for standardized series, averaged over variables and time periods) in the training, validation and testing sub-samples are given; predictions are for multiple steps ahead in the latter two sub-samples. Additional technical details about the implementation are provided in the main text.

Table 2: Vector Auto-Regression, Prediction Results

L	Training	Validation	Training&Validation	Testing
4	0.5128	4.2544		4.9624
11			0.5372	5.7299

Notes: The results are presented, in rows, for VARs with lag lengths $L = 4$ (selected basing on validation sample performance) and $L = 11$ (selected basing on AIC). Estimation was done by OLS. MSEs (calculated for standardized series, averaged over variables and time periods) in the training, validation and testing as well as combined training & validation sub-samples are given; predictions are for multiple steps ahead in the validation and testing sub-samples. Additional technical details about the implementation are provided in the main text.

- $(5 + 50) = 55$ parameters in the visible output layer.

The validation sample predictive fit is 5.1148 (multiple-steps-ahead, accuracy measured in MSE terms). This is worse than in some of the alternative trained architectures; the minimum is 4.7265 for a 4-layered NN with 400 units in each. However, given that our validation sample is very small due to data limitations, it is impossible to tell apart an underperformance of the specific NN instance and a peculiarity of the chosen validation data sample.¹²

The testing sample was not available for our preferred NN at the training stage: its predictive fit ended up being 4.0858 there. Many alternative NNs performed even better, with the minimum MSE being 3.8704 for a 1-layered NN with 300 units; but the testing sample and the corresponding predictive performances became known only ex-post.

We are going to compare our best NN with the benchmark model, Vector Auto-Regression. Predictive performance on the validation sample suggested using 4 lags, i.e. VAR(4). Such specification comprises 105 parameters in total (refer to equation 6): $(5+25+25+25+25)$. As can be seen from Table 2, VAR model’s in-sample performance dominates that of our preferred NN: the training sample MSE is 0.5128, and the validation sample MSE is 4.2544. However, VAR underperforms out-of-sample: for the test data, its MSE of 4.9624 is about 20% worse. (For comparison, Akaike Information Criterion suggested using 11 lags, the resulting VAR(11) model in Table 2 performs poorly out-of-sample, its test-data MSE is 5.7299; we prefer the VAR(4) specification as a benchmark.)

Let us zoom closer into two models’ predictions on Figure 1. Neither of the two captures high-frequency movements, instead they are trying to pin down the movements in

¹²That is why we relied on the—longer—training sample’s fit for architecture choice.

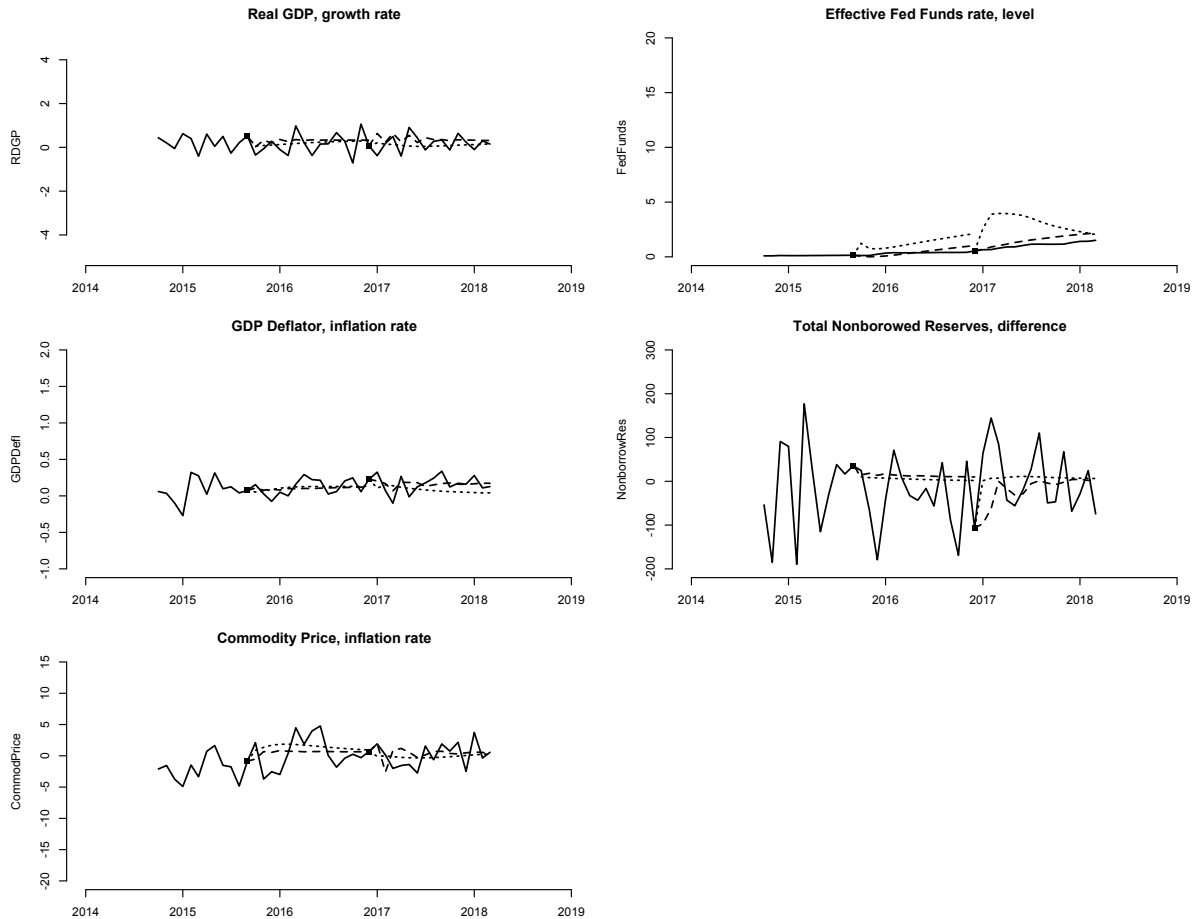


Figure 1: Predictive performance.

Multiple-steps-ahead predictions of the MRNN (dotted) and VAR (dashed) models over the testing sample: first, period 2015:M10 to 2016:M12, and second, period 2017:M1 to 2018:M3; with the starting points, 2015:M9 and 2016:M12, marked (by solid squares); together with the historical data from 2014:M10 to 2018:M3 (solid).

conditional means. Failure to achieve the former, much more challenging task should not be surprising: the time series in question are inherently noisy (just think of unpredictable weather conditions), they are affected by many other factors beyond the 5-variable economic system that is being modeled (think of housing market dynamics or global economic environment).

One point to keep in mind is that for most practical purposes, what matters is the cumulative forecast over several periods. In such a situation, much of the high-frequency noise cancels out. Moreover, even a small improvement in prediction accuracy can be very valuable: say, an improvement of merely 0.1 percentage point per month produces more than 1 percentage point improvement in the case of GDP growth forecast on a 12-months horizon.

Another point concerns our standardization convention. We conduct the modeling in

terms of standardized time series, each having the variance of 1, and weight them equally in the mean-squared loss function. So, in the original scaling some of the forecasts on Figure 1 may look farther off the target than they are in the standard-deviations space that is important from the modeling perspective. E.g., NN’s Fed Funds rate predictions are not as bad as they may seem in the original scaling. For the reference, the standard deviations of our 5 variables [Real GDP growth, GDP Deflator inflation, Commodity Price inflation, Fed Funds rate, Nonborrowed Reserves change] are [0.91, 0.22, 2.76, 3.53, 26.16]; thus the magnitudes of the plotted prediction errors in Fed Funds rate and Nonborrowed Reserves change are overblown, and those of GDP Deflator inflation are underemphasized.¹³

Overall, the take-away is that, at least in our exercise, MRNN outperformed VAR in out-of-sample prediction (and, thus, in generalization).

4.2.2 Steady states

Now we briefly investigate the limiting behavior of the system, which reveals some robust features of the data and the typical scenarios in the evolution of economic variables. In other words, how does the economy look on average, across many time periods?

Technically, a steady state (an equilibrium, or a “fixed point”) of a multivariate time-series system is defined as some subset of the space that the system maps back to itself. A stable steady state (attractive fixed point, or a “fixed-point attractor”) is a steady state that, even after some small perturbation, the system will again converge to (instead of diverging to somewhere else). Then, a region of convergence (“basin of attraction”) is a neighborhood of a stable steady state that maps the outputs from its inputs in the direction toward, and will eventually reach, this particular steady state. The stable steady states characterize the asymptotic behavior of the economic system.

In the case of VAR, such investigation can be done by solving the system analytically. For MRNN, the easiest approach is to iterate the system forward till convergence.

Table 3 contains the results of the corresponding calculations. Naturally, VAR steady state is close to the sample mean. However, MRNN has two stable steady states: first, characterized by a combination of low inflation and interest rates (as well as high growth rate); and second, with—relatively—high inflation and interest rates (as well as low growth rate). Mathematically, existence of several steady states is a common situation in non-linear time-series models, e.g. see Tong (1990). Intuitively, this result reveals that the NN has learned about the presence of two different macroeconomic regimes, and has done so in an unsupervised manner.

The region of convergence for VAR is the whole real hyperplane \mathbb{R}^K . For MRNN, the region of convergence is a K -dimensional object that is generally cumbersome to inspect in detail; but this is a straightforward task for a handful of dimensions conditionally on

¹³Interestingly, as can be seen in Figure 1, on the basis of past history MRNN predicts a more aggressive tightening of the Fed’s policy rate, and, correspondingly, lower inflation rates than we actually saw happening lately.

Table 3: Steady States

Model	RGDP	GDPDefl	CommodPrice	FedFunds	NonborrowRes
$\bar{\mathbf{y}}$	0.252	0.285	0.263	5.484	2.758
MRNN, low regime	0.314	0.195	0.217	4.023	0.552
MRNN, high regime	0.191	0.299	-0.179	8.816	-0.609
VAR	0.252	0.279	0.284	5.260	3.480

Notes: Stable steady state values are presented, in rows, for MRNN (low GDP Deflator inflation and low Fed Funds rates, high GDP Deflator inflation and high Fed Funds rates); as well as for VAR with lag length $L = 4$; also, $\bar{\mathbf{y}}$ provides the sample averages, for reference. Variables, in columns, are Real GDP growth, GDP Deflator inflation, Commodity Price inflation (all month-over-month, in percents); Fed Funds rate (level, in percents); Nonborrowed Reserves change (monthly difference, in billions of US dollars).

the rest. For example, take the low inflation and interest rates steady state: freezing all of the variables at their steady state values, we find that an acceleration of GDP Deflator inflation from 0.195% to 0.272% or more sets the system on a trajectory that diverges to another, high inflation and interest rates steady state.

4.2.3 Impulse responses

Time-series models can also be used for measuring system reactions to shocks, as well as to conduct hypothetical policy simulations. For example, if commodity prices (think of oil) jump sharply up, what would happen to other economic variables? Or how would the economy at a particular macroeconomic juncture (think of deep recession or Zero Lower Bound) fare when the Fed Funds rate is raised and fixed at some specified level?

Technically, impulse responses of a multivariate time-series system are defined as movements (responses) of all variables once the equilibrated system receives a single-period shock (impulse) to one of the variables. A single-period shock is called transitory, a shock that persists for an infinite number of periods is called permanent.¹⁴

Unconditionally: The leading case that we consider is when the system is at its steady state immediately before receiving the shock.

In VAR, the process works as follows. Given the covarying nature of the system variables \mathbf{y} , and thus of the shocks to these variables $\boldsymbol{\xi}$, a correct way to generate a single shock has to account for the covariance structure. Perhaps the simplest way of doing this is using a Cholesky decomposition of the shocks' variance-covariance matrix: $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^\top$, where \mathbf{L} is the resulting lower-triangular matrix. (Technically, this allows to identify contemporaneous links between variables in our reduced-form VAR using recursive identification restrictions.) As a result, it holds that $\boldsymbol{\xi} = \mathbf{L}\boldsymbol{\varepsilon}$, given some fundamental uncorrelated shock $\boldsymbol{\varepsilon}$ that has an identity variance-covariance matrix \mathbf{I} .

Thus, a transitory shock to variable k at impulse time τ is cleanly injected by (i) perturbing the k -th element of $\boldsymbol{\varepsilon}$: $\boldsymbol{\varepsilon}_\tau := \mathbf{e}_k$, where \mathbf{e}_k is a column vector with, say, 1 in row k and 0 in all other rows; and then (ii) adding the manufactured shock $\boldsymbol{\xi}_\tau := \mathbf{L}\boldsymbol{\varepsilon}_\tau$ to the previously equilibrated system: $\mathbf{y}_\tau := \bar{\mathbf{y}} + \boldsymbol{\xi}_\tau$, where $\bar{\mathbf{y}}$ is a steady-state value. Instantaneously in the same period τ , the shock percolates to the variables with non-zero elements of lower triangular matrix \mathbf{L} ; from period $\tau + 1$ onward, the whole system is affected according to \mathbf{y} 's law of motion. Injecting a permanent shock is more intricate, and we implement this simply by generating a sequence of unanticipated¹⁵ transitory shocks: each of the latter is injected in exactly the same way as before, except that the sizes of shocks are dynamically rescaled to account for the persistence already generated by the

¹⁴In this paper we ignore potential asymmetry of responses and thus do not examine separately impulses in the opposite directions.

¹⁵Our setting does not allow for an explicit permanent shock, and what we end up doing may be interpreted as a permanent shock under weak credibility of the central bank.

law of motion endogenously, that is $\boldsymbol{\varepsilon}_{\tau+1} := \boldsymbol{\varepsilon}_{\tau} - \mathbf{A}_1 \boldsymbol{\varepsilon}_{\tau}$, and so on for periods after $\tau + 1$.

Unfortunately, in LSTM MRNN a similarly straightforward approach as above is not applicable. Given that LSTM networks comprise not only observed variables \mathbf{y} , but also unobserved state variables \mathbf{c} and \mathbf{h} , the shocks should perturb both the observed as well as the unobserved variables.¹⁶ Ideally, this should be done by a careful injection of shocks to the state variables (and/or having imposed restrictions on the relevant connection parameters). But this is a challenging task given the complexity of a typical ANN’s network of hidden units and connections. For instance, generally there is no single state variable devoted to the variable that is to be shocked, the relevant elements can be scattered anywhere between and within the hidden units.

One indirect and naïve way that seems to work in practice is as follows. The transitory shock $\boldsymbol{\xi}_{\tau}$ is manufactured using Cholesky decomposition as above. During the usual “warmup session” that is necessary for initialization of the trained LSTM before any usage (and in our case is carried out by feeding the NN with variables’ steady-state values), in the last 12 periods of the session the shock $\boldsymbol{\xi}$ is added on top of the usual “warmup” inputs, thus shifting the corresponding state variables in the appropriate direction. Then, at impulse time τ , the shock is added, $\mathbf{y}_{\tau} := \bar{\mathbf{y}} + \boldsymbol{\xi}_{\tau}$, and the system evolves as above. Given this procedure of injecting a transitory shock, for a permanent shock we repeat its implementation from the VAR setting, except that now the persistence term is more complicated, i.e. $\boldsymbol{\varepsilon}_{\tau+1} := \boldsymbol{\varepsilon}_{\tau} - \mathbf{L}^{-1}(\hat{\mathbf{y}}_{\tau+1} - \bar{\mathbf{y}})$.

Reactions of the VAR and two MRNN models to a 1 percentage point (i.e., 100 basis points) increase in the Fed Funds rate are presented in Figures 2, 3 and 4, respectively.¹⁷ The following observations can be made:¹⁸

- (i) For the output, economic intuition suggests contraction/slowdown in response to transitory Fed Funds rate increase (monetary contraction reduces nominal aggregate demand, affecting not only nominal, but also real quantities); and either no effect or contraction/slowdown after a permanent increase (depending on how efficiently the economy adjusts to a nominal shock). VAR implies a growth acceleration after both types of shocks in the short run, and a slowdown after the permanent shock in the long run. MRNN (for either regime) implies a short-run growth slowdown for both types of shocks, as well as a long-run slowdown after the permanent shock.
- (ii) For the output price inflation rate, economic intuition suggests deceleration in response to transitory Fed Funds rate increase (monetary contraction reduces nominal

¹⁶Basically, what should be shifted is the shocked variable’s conditional mean, which is (loosely speaking) determined by the corresponding unobserved state variables. Otherwise, in our practice the shocked variable and the whole system just reverted to the unconditional mean almost immediately.

¹⁷Note that given the existence of more than 1 stable steady state in MRNN, the impulse responses offered by MRNN are in certain sense more “local” than the “global” responses provided by VAR.

¹⁸Henceforth, we use the wording “short run” and “long run” in terms of presented impulse response horizons, rather than in the sense of, correspondingly, transitory and permanent effects.

aggregate demand, initiating deflationary pressure); but acceleration after a permanent increase in the long run (according to Fisherian views, real interest rate is determined by economic fundamentals and can not be easily manipulated, higher nominal interest rate ultimately translates into higher inflation rate) and possibly even in the short run (according to Neo-Fisherian views). VAR implies an acceleration of output inflation after both types of shock in the short run, as well as after the permanent shock in the long run. MRNN (low regime) implies a short-run inflation deceleration after both types of shock, as well as a long-run one after the transitory shock; MRNN (high regime) implies a short-run inflation acceleration after both types of shock, as well as a long-run one after the transitory shock.

- (iii) For commodity prices, economic intuition suggests contraction/deceleration in response to a Fed funds rate increase (tighter monetary supply increases the relative price of currency in terms of goods). VAR implies some wobbling of the commodity inflation after both types of shock in the short-run, and a deceleration after the permanent shock in the long-run. MRNN (for either regime) implies a short-run commodity inflation deceleration after both types of shock, as well as a long-run one after the permanent shock.
- (iv) For Fed Funds rate, a stable time-series system should deliver gradual vanishing of its response to the initial impulse (as long as the shock is transitory). VAR implies a gradual reversion of Fed Funds rate to its mean after a transitory shock. MRNN (for either regime) implies a gradual mean-reversion after the transitory shock, albeit with some undershooting along the way.
- (v) For the nonborrowed reserves that banks hold with the central bank, economic intuition suggests their contraction/slowdown in response to a Fed Funds rate increase. VAR implies a slowdown in reserve accumulation after both types of shock in the short run, as well as after the permanent shock in the long run. MRNN (low regime) implies a short-run mixed dynamic after a transitory shock and a short-run slowdown after the permanent shock, as well as a long-run slowdown after the permanent shock. MRNN (high regime) implies a short-run slowdown after both types of shock, as well as a long-run one after the permanent shock.

Apart from relatively mild quantitative differences, the key qualitative differences between MRNN and VAR demonstrated in this exercise concern the responses of Real GDP and GDP Price Deflator growth rates. The effect of monetary contraction on output growth is unambiguously negative in MRNN model, in either of its two economic regimes. But the effect is positive before turning negative in VAR model. The effect of monetary tightening on inflation is negative in MRNN's regime characterized by low inflation and interest rates, and it is mixed in MRNN's regime characterized by high inflation and interest rates. However, the effect is unambiguously positive in VAR model. Arguably,

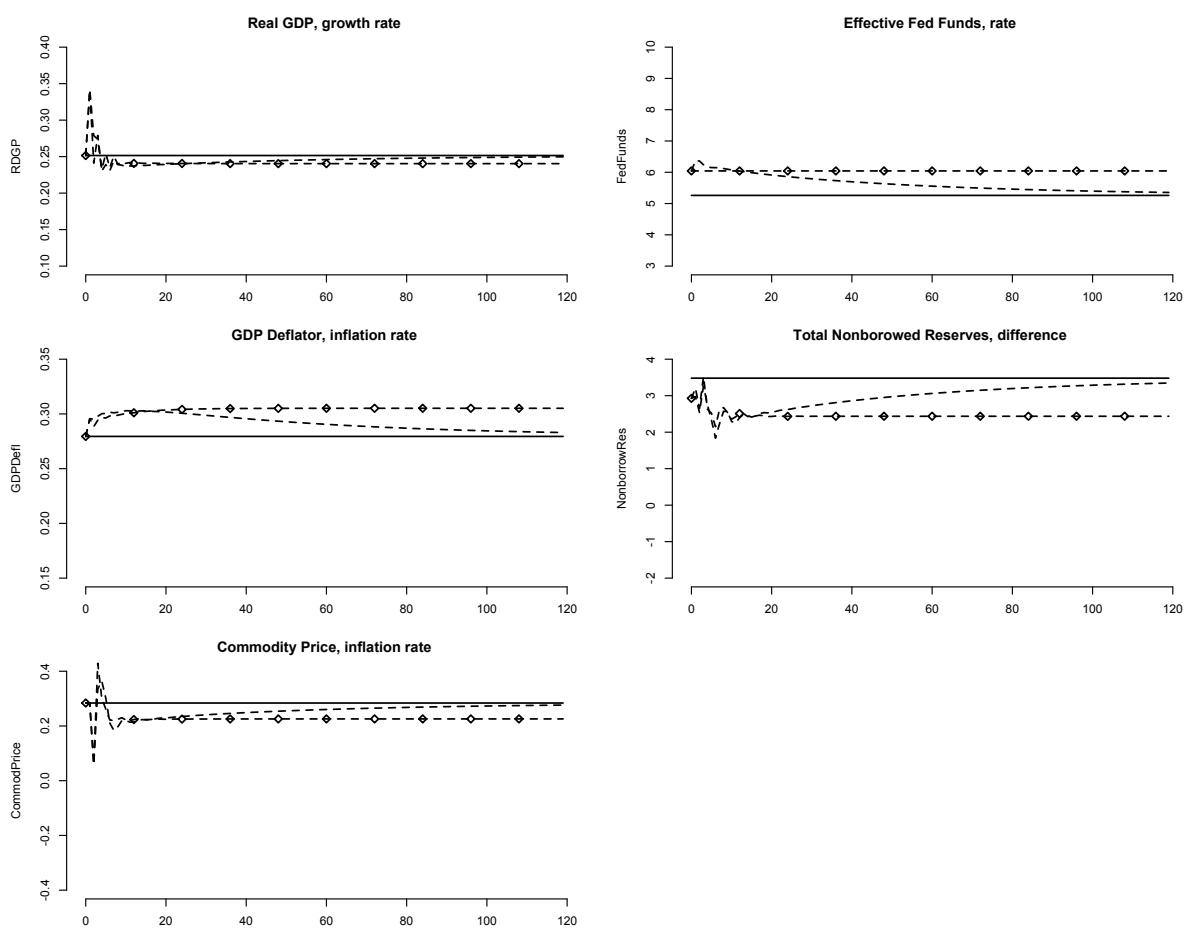


Figure 2: Impulse response functions: VAR.

Reactions in the VAR model to transitory (dashed) and permanent (dashed with diamonds) shocks of +1 p.p. in the Effective Fed Funds rate variable, together with the estimation sample means (solid).

Units on the horizontal axis: months.

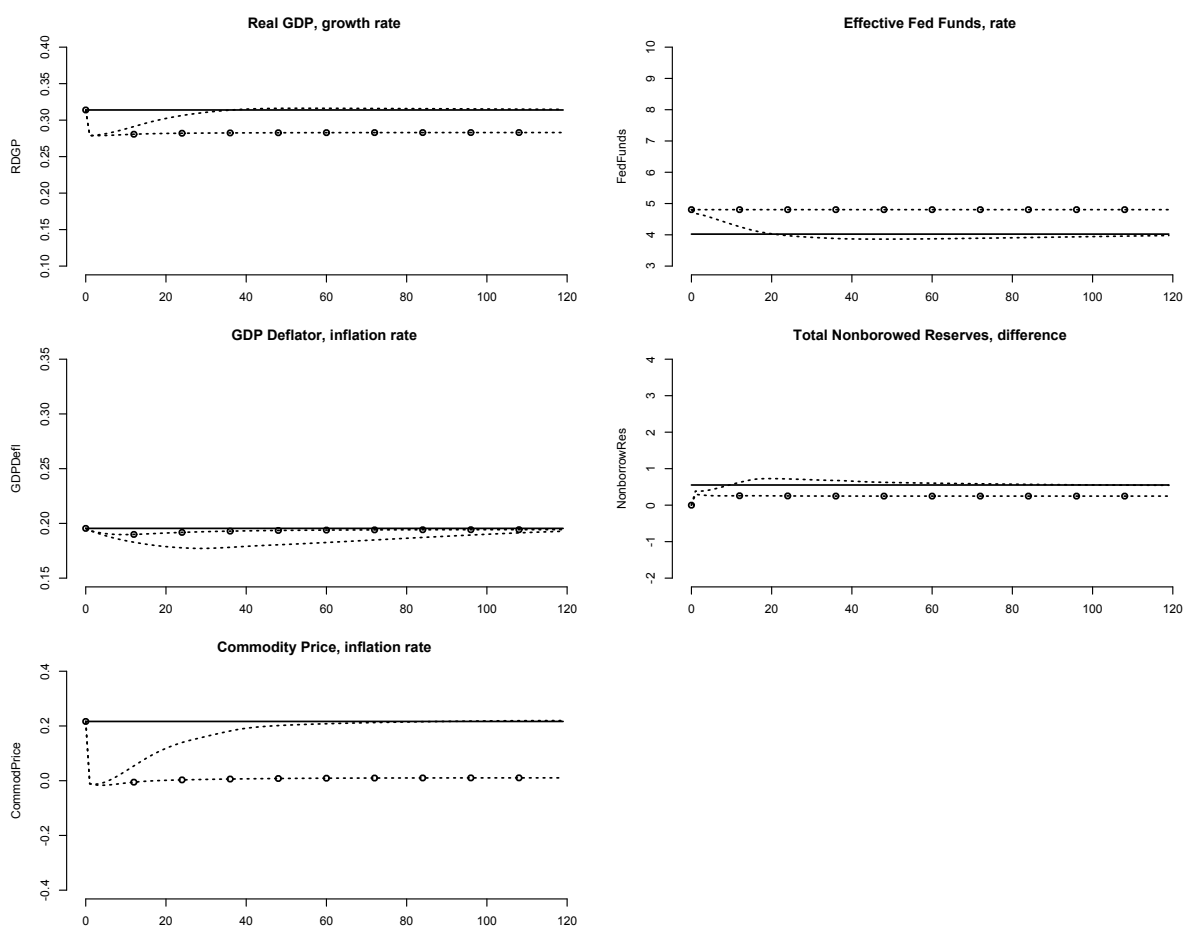


Figure 3: Impulse response functions: MRNN, low regime.

Reactions in the MRNN model to transitory (dotted) and permanent (dotted with circles) shocks of +1 p.p. in the Effective Fed Funds rate variable, together with the low regime's steady state values (solid).

Units on the horizontal axis: months.

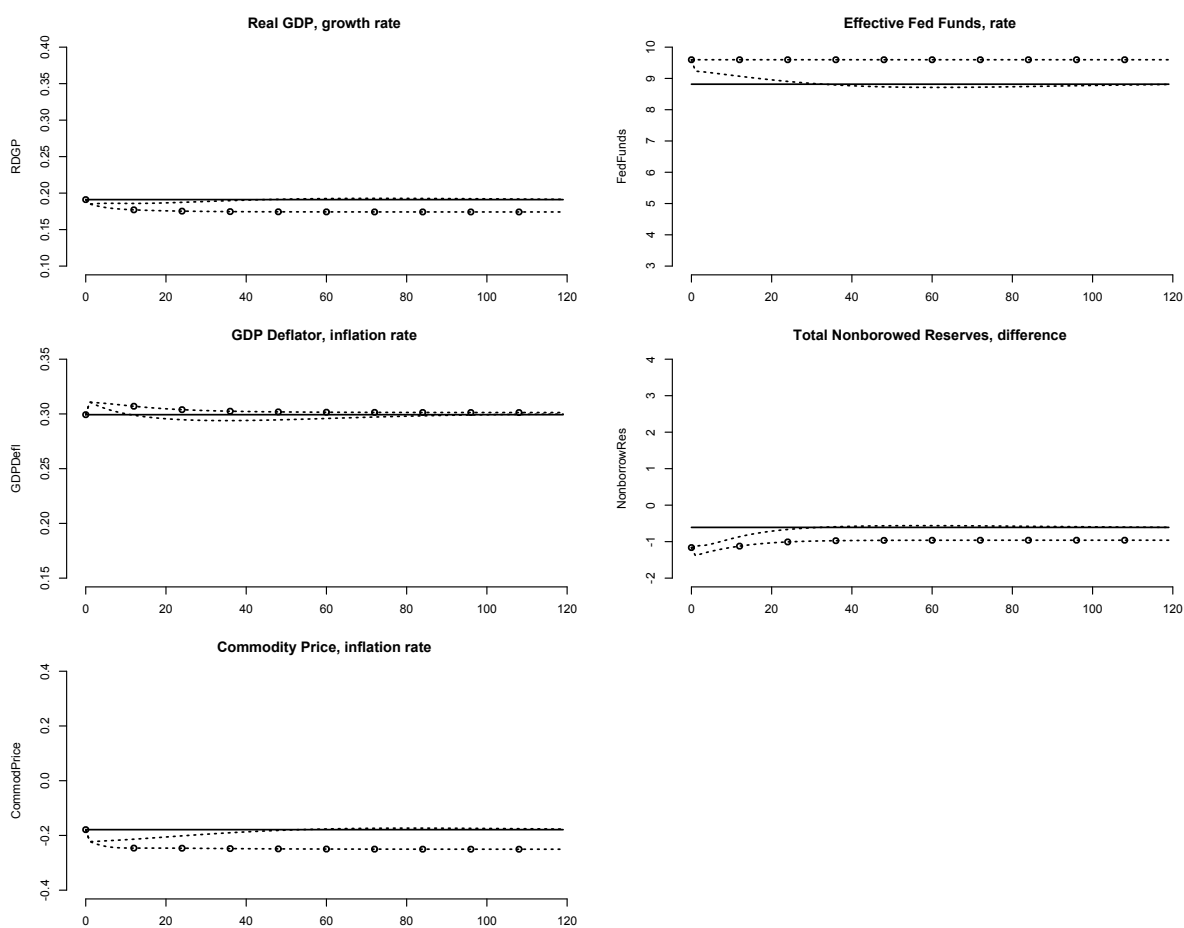


Figure 4: Impulse response functions: MRNN, high regime.

Reactions in the MRNN model to transitory (dotted) and permanent (dotted with circles) shocks of +1 p.p. in the Effective Fed Funds rate variable, together with the high regime's steady state values (solid). Units on the horizontal axis: months.

the system responses are more plausible from economic theory perspective in the case of MRNN.

Conventional vs. (Neo-) Fisherian views in VAR context: While the effect of monetary policy on output growth is relatively well understood in economic theory, its effect on price inflation is more controversial.

Conventional economic theory predicts that Fed Funds rate increase leads to lower inflation (see e.g. Christiano et al., 2005), even though not necessarily materializing in the short run.

The so-called “price puzzle”, implying positive response of inflation to monetary policy rates, has been first noted in a VAR model due to Sims (1992). Apart from the effect of some omitted variables, a sensible interpretation of this finding is that simple, theoretically unrestricted VARs “confuse” correlation with causation. Historically, a high rate of inflation has been associated with high nominal interest rates, and while higher inflation indeed causes a central bank to increase the policy interest rate (one direction of causality), such an increase itself causes inflation rate to subside (another, opposite direction of causality).¹⁹ With the aim of clarifying contemporaneous causal links between variables in VARs, a large literature on structural identification restrictions has been born (leading to so-called structural VARs, which “decompose” reduced-form VARs by applying identification schemes that are mainly based on economic theory as opposed to relatively atheoretical and “mechanistic” schemes such as recursive identification or identification by heteroscedasticity). It includes approaches based on formal theoretical models (see e.g. King et al., 1991; Galí, 1992; Bernanke and Mihov, 1998) as well as those based on more agnostic theoretical assumptions (for example, sign restrictions, as in Uhlig, 2005; Antolín-Díaz and Rubio-Ramírez, 2018; Uribe; 2018).

On the other hand, Fisherian arguments predict that permanent Fed Funds rate increase leads to higher inflation in the long run, with the Neo-Fisherian thinking (Cochrane, 2018; Williamson, 2016) suggesting that higher inflation may appear even in the short run.

Interpreting the two regimes: Now, let us get back to the issue of two stable steady states that the MRNN has identified earlier (in §4.2.2).

Looking at the differences in impulse response functions in Figures 3 and 4, the regime with low inflation and interest rates (as well as high growth) demonstrates a disinflationary effect of monetary contraction and is more in line with the conventional intuition. On

¹⁹Identifying causal relations and the notion of causality itself is a profound question, we do not even attempt here to give it due justice. Two perspectives most relevant to social sciences are presented in Rubin (2005) or Morgan and Winship (2007) (potential outcomes framework) as well as in Pearl (2000) (causal calculus formulated in terms of Directed Acyclic Graphs). In natural sciences, experimental findings such as—taking just a few—Goswami et al. (2018) or Ringbauer et al. (2016) pose further theoretical challenges (for example, see Brukner, 2014; Allen et al., 2017).

the other hand, the regime with high inflation and interest rates (as well as low growth) exhibits a less potent disinflationary response, if at all, thus corroborating a more Fisherian view.

Looking at the differences in the quantitative features of the two regimes in Table 3, we notice that real growth and nominal inflation layers, though combining differently in each regime, add up to the same annualized 6% (0.5 percentage points per month) level in both. Nevertheless, the Fed Funds rate averages just 4.0% in the low inflation and interest rates regime, but as much as 8.8% in the high inflation and interest rates one.

In understanding the nature of these “good” and “bad” equilibria, the numbers from Table 3 may reflect the differences in economic structure (perhaps the costs of inflation play a role), or just a prevailing phase of the business cycle/shape of the yield curve in each of the two regimes. However, the reactions from Figures 3 and 4 indicate that issues of coordination between economic agents, central bank credibility, anchoring of inflation expectations are probably also part of the picture.

Conditionally (Zero Lower Bound): Another case that we examine is the impulse responses when the system is placed into specific economic environment, with the situation around Zero Lower Bound being particularly interesting. In contrast to VAR, LSTM MRNN may be especially useful in this case, given the latter’s ability to capture state-dependent dynamics.

We pick January 2011 to January 2012 as the time interval when the Federal Reserve’s main policy instrument, Fed Funds rate, was at historically low levels, and was constrained from below by the interest rate of zero that promises no additional return to the lender of funds (i.e., was at Zero Lower Bound). After initializing MRNN model by historical data (January to December 2011) rather than by the steady state values, we add the shock(s) on top of historical data immediately afterwards (January 2012) and compute MRNN’s impulse response(s) the same way as before. The results are presented in Figure 5.

After the transitory shock to Fed Funds rate, it gradually moves toward the steady state (corresponding to low regime). The figure shows a mixed effect on output growth in the short run, and no effect in the long run; it also shows a mixed effect on inflation in the short run, which then accelerates before reverting to the steady-state level in the long-run.

After the permanent shock, Fed Funds rate stays at the low level. The figure shows a short-run mixed effect on output growth, with a long-run (slight) acceleration; it also shows a short-run mixed effect on inflation, and its long-run (strong) acceleration.

Impulse responses to the permanent shock exhibited by MRNN in low regime from Figure 3 and by MRNN around ZLB here in Figure 5 provide an especially clean comparison. In the latter case, both the output growth is relatively stronger (and exceeds the steady-state mean) as well as the inflation rate is relatively higher (and also exceeds the steady-state mean).

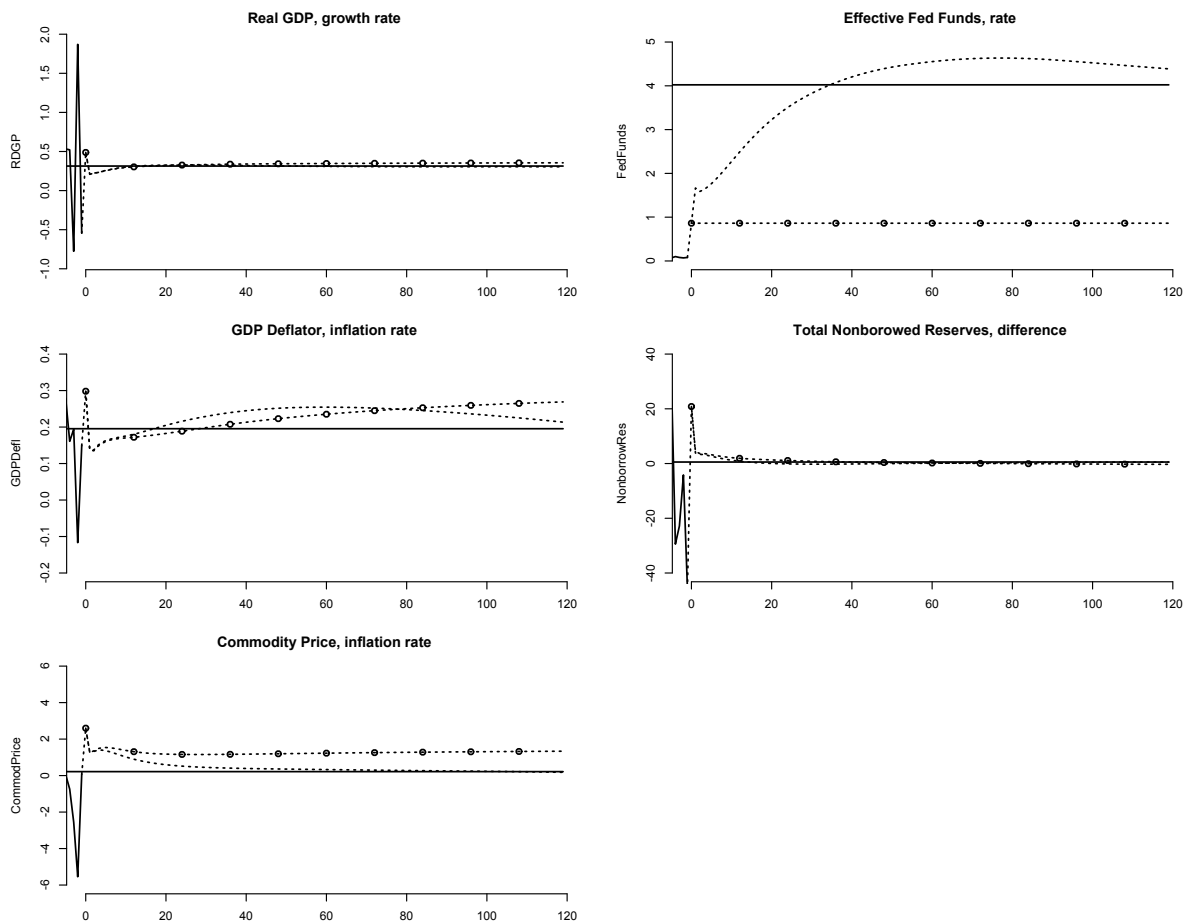


Figure 5: Impulse response functions: MRNN, at Zero Lower Bound.

Reactions in the MRNN model to transitory (dotted) and permanent (dotted with circles) shocks of +1 p.p. in the Effective Fed Funds rate variable, together with the low regime's steady state values (solid horizontal) and the historical data from 2011:M1 to 2011:M12 (solid). Units on the horizontal axis: months.

4.2.4 Data augmentation

As a robustness check, in particular to address possible concerns about relying on a very limited data set, we attempt a data augmentation exercise.

Since in our modeling approach we abstain from imposing any structure on our data, the augmentation strategy we chose is noise contamination. Specifically, we (i) divide the existing (standardized) time series into blocks of 50 consecutive observations; (ii) copy the data set 10 times; (iii) randomly shuffle the blocks in the enlarged data set; (iv) add Gaussian noise (with zero mean and a variance-covariance observed in the training sample) scaled by 0.1; (v) additionally paste the last block of the (uncontaminated) training data set at the end of the time series. In the end, we extend the training data set from 650 to 6550 observations, leaving the validation and test samples untouched.

The MRNN on augmented data set was trained exactly the same way as before (and also for 20 epochs). When we applied the trained model to the uncontaminated data set, the measures of fit on all three sub-samples underperformed those of our original MRNN: 0.6465 vs. 0.6311 in the training sample, 5.1902 vs. 5.1148 in the validation sample, and 4.4893 vs. 4.0858 in the testing sample.

Our data augmentation approach did not seem to offer any advantage over using just the existing data as is. However, a different approach may prove to be more fruitful, hence verification of different augmentation settings or perhaps even trying a more structural augmentation strategy is still a worthwhile exercise.

5 Conclusion

In modeling half a century of US economic time-series data, Multivariate Recurrent Neural Networks outperformed mainstream Vector Auto-Regressions in terms of forecasting accuracy and interpretability of their results. Higher precision of its out-of-sample predictions suggests that MRNN acquires stronger generalization ability. Also, MRNN discovers different macroeconomic regimes under a “hands-off” procedure, without any special modeler’s intervention. Moreover, MRNN appears to learn theoretically plausible causal effects from the raw data, without the need to impose any structural identification restrictions beyond a simple recursive ordering of variables. Arguably, non-linear architecture of NNs and the prominence of predictive criterion in their training, as well as Long Short-Term Memory MRNN’s capability to account for conditional dynamics of the modeled system’s state variables make them well equipped for such tasks.

A Data inputs

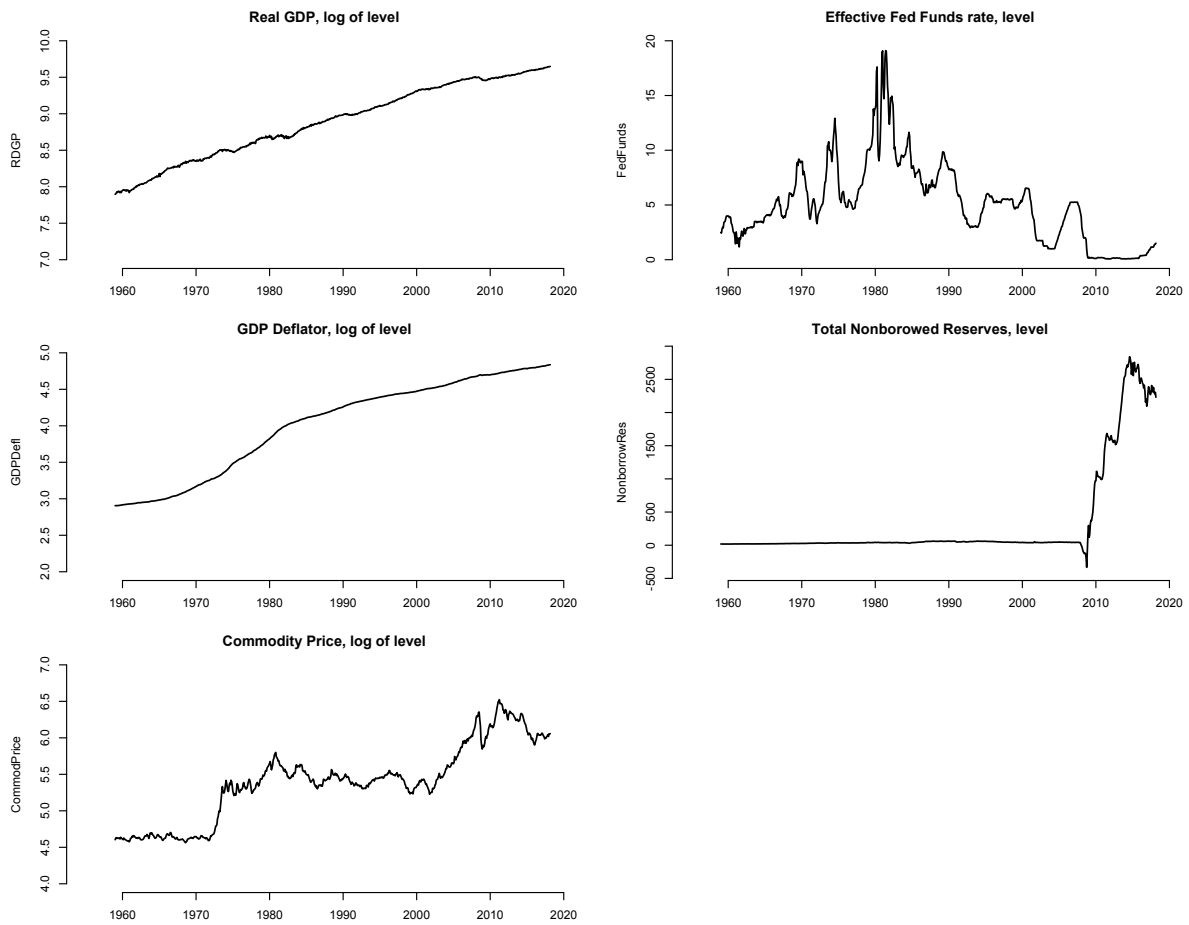


Figure 6: Plots of the initial data inputs.

Variables included: logarithm of Real GDP, logarithm of GDP Price Deflator, logarithm of Commodity Price Index, Effective Fed Funds Rate and Total Nonborrowed Reserves.

Time period covered: 1959:M1 to 2018:M03.

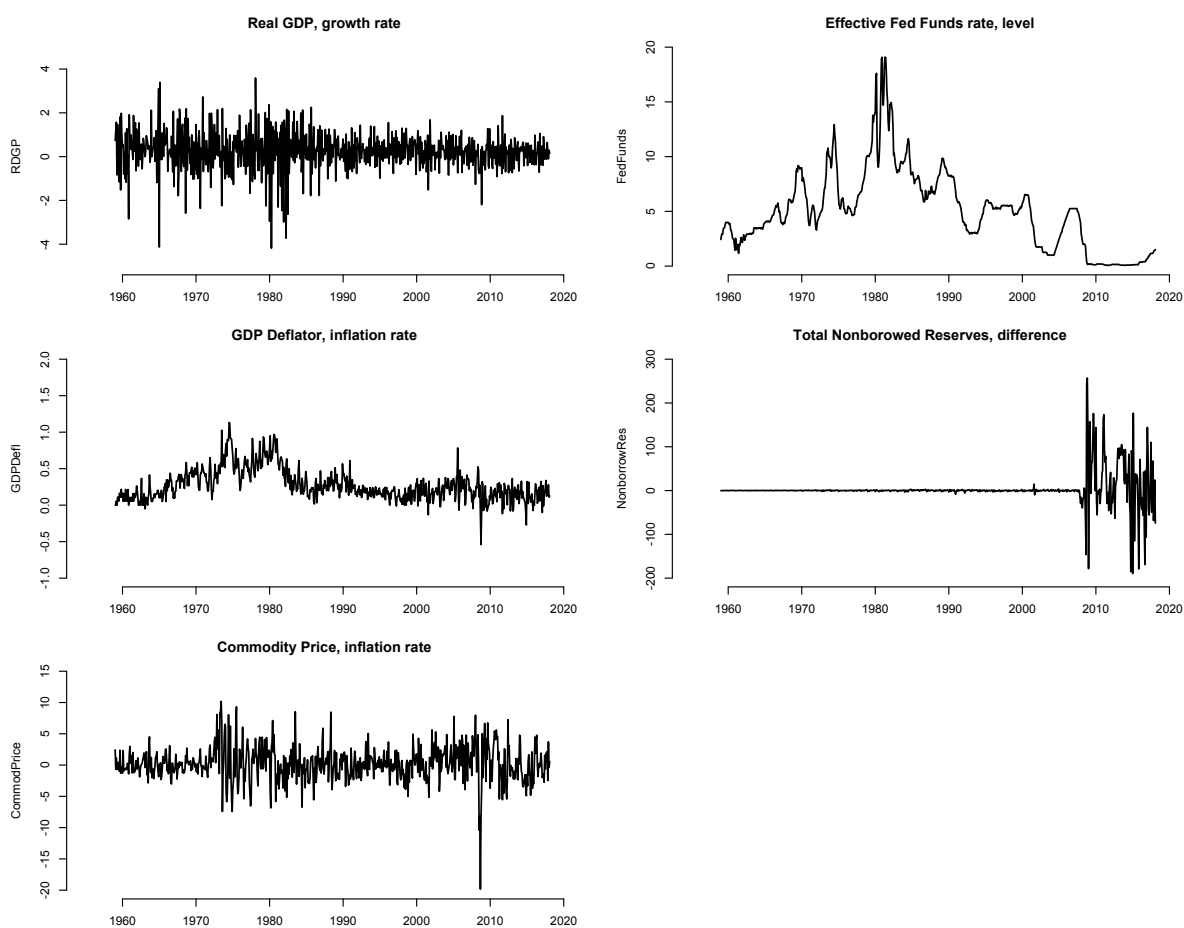


Figure 7: Plots of the transformed data inputs.

Variables included: Real GDP MoM growth rate, GDP Price Deflator MoM growth rate, Commodity Price Index MoM growth rate, Effective Fed Funds Rate and Total Nonborrowed Reserves monthly difference.

Time period covered: 1959:M2 to 2018:M03.

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