Log, Stock and Two Simple Lotteries. Supplement I*

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Abstract

These supplemental materials include extensions and details on the model used in the main text, as well as on its solution. An abbreviated overview of the related literature is also offered. Proofs for the lemmas, propositions, theorems and corollaries stated in the main text are collected here too.

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A.1 Investment portfolio choice problem

The consumer-investor is interested in solving the following consumption and portfolio choice problem, $\mathcal{P}_{\mathcal{Q}}$:

$$\max_{\{C_s,\{q_{0,s},\boldsymbol{q}_s\}\}_t^{\infty}} \mathrm{E}_t^g\left[\sum_{s=t}^{\infty}\beta^{s-t}u(C_s)\right] = \int_{\mathbb{R}_+^K}\sum_{s=t}^{\infty}\beta^{s-t}u(C_s)\,g_D(\boldsymbol{D}_s|\boldsymbol{D}_{s-1})\,d\boldsymbol{D}_s \qquad \{\mathcal{P}_{\mathcal{Q}}\}$$

subject to a sequence of budget constraints

$$C_s + P_{0,s}q_{0,s} + \boldsymbol{P}_s^{\intercal}\boldsymbol{q}_s = q_{0,s-1} + (\boldsymbol{P}_s+\boldsymbol{D}_s)^{\intercal}\boldsymbol{q}_{s-1}, \qquad \forall s \geq t,$$

control variables' domain restrictions $C_s, \{q_{0,s}, \boldsymbol{q}_s\} \in \mathbb{R}_+ \times \mathbb{R}^{K+1}, \forall s \geq t$, as well as the no-Ponzi-schemes constraint, also listing here the usual transversality condition for optimality,

$$\lim_{T\to\infty}\left\{\left(\prod_{s=t}^{T}P_{0,s}\right)q_{0,T-1} + \mathbf{1}^{\intercal}\left(\prod_{s=t}^{T}\mathrm{diag}(\boldsymbol{P}_s+\boldsymbol{D}_s)^{-1}\mathrm{diag}(\boldsymbol{P}_s)\right)\boldsymbol{q}_{T-1}\right\} \geq 0 \quad \text{a.s. (under } g_D\text{)},$$

$$\lim_{s\to\infty}\mathrm{E}_t^g[\beta^{s-t}u'(C_s)(P_{0,s}q_{0,s}+\boldsymbol{P}_s^{\intercal}\boldsymbol{q}_s)] = 0;$$

with $u(C_s) = C_s^{1-\gamma}/(1-\gamma)$, and where

$$g_D(\boldsymbol{D}_{s+1}|\boldsymbol{D}_s) \text{ is given,} \qquad \forall s \geq t.$$

(Alternatively, the no-Ponzi-schemes constraint and the transversality condition can be replaced with a compact domain for admissible control variables that covers the borrowing/short-selling and the asset supply limits.)

In words, the representative agent would like to choose stochastic consumption and investment plans that maximize an expected discounted sum of per-period utilities and at the same time satisfy the sequence of budget constraints (as well as the technical conditions ruling out pathological and ensuring valid solutions). The expectation is taken with respect to a given objective probability density function that defines the distribution of the stochastic fruit-dividends.

This is a standard dynamic programming problem. The state variables are $\{q_{0,t-1}, \boldsymbol{q}_{t-1}\}$ and $\boldsymbol{D}_t$. Denote the maximum value function as $v^{\sharp}(\{q_{0,t-1}, \boldsymbol{q}_{t-1}\}, \boldsymbol{D}_t)$. The corresponding Bellman equation is then:

$$v^{\sharp}(\{q_{0,t-1},\boldsymbol{q}_{t-1}\},\boldsymbol{D}_t) = \max_{C_t,\{q_{0,t},\boldsymbol{q}_t\}}\left\{u(C_t) + \beta\mathrm{E}_t^g\left[v^{\sharp}(\{q_{0,t},\boldsymbol{q}_t\},\boldsymbol{D}_{t+1})\right]\right\}$$

subject to

$$C_t + P_{0,t}q_{0,t} + \boldsymbol{P}_t^{\intercal}\boldsymbol{q}_t = q_{0,t-1} + (\boldsymbol{P}_t+\boldsymbol{D}_t)^{\intercal}\boldsymbol{q}_{t-1},$$

domain restriction $C_t, \{q_{0,t}, \boldsymbol{q}_t\} \in \mathbb{R}_+ \times \mathbb{R}^{K+1}$, as well as the same no-Ponzi-schemes condition; also with the same utlity function specification, and where

$$g_D(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t) \text{ is given.}$$

A.2 Feasible investment portfolio choice problem

A feasible version of the consumption and portfolio choice problem, $\mathcal{P}_{\mathcal{QI}}$, is formulated as follows:

$$\max_{\{C_s,\{q_{0,s},\boldsymbol{q}_s\}\}_t^\infty} \mathrm{E}_t^h\left[\sum_{s=t}^{\infty}\beta^{s-t}u(C_s)\right] = \int_{\mathbb{R}_+^K}\sum_{s=t}^{\infty}\beta^{s-t}u(C_s)\,h_D(\hat{\boldsymbol{D}}_s|\hat{\boldsymbol{D}}_{s-1})\,d\hat{\boldsymbol{D}}_s \qquad \{\mathcal{P}_{\mathcal{QI}}\}$$

subject to a sequence of budget constraints

$$C_s + P_{0,s}q_{0,s} + \boldsymbol{P}_s^\intercal\boldsymbol{q}_s = q_{0,s-1} + (\boldsymbol{P}_s + \hat{\boldsymbol{D}}_s)^\intercal\boldsymbol{q}_{s-1}, \qquad \forall s \geq t,$$

control variables' domain restrictions $C_s, \{q_{0,s}, \boldsymbol{q}_s\} \in \mathbb{R}_+ \times \mathbb{R}^{K+1}, \forall s \geq t$, as well as the no-Ponzi-schemes constraint, also listing here the usual transversality condition for optimality,

$$\lim_{T\to\infty}\left\{\left(\prod_{s=t}^{T}P_{0,s}\right)q_{0,T-1} + \mathbf{1}^\intercal\left(\prod_{s=t}^{T}\mathrm{diag}(\boldsymbol{P}_s+\hat{\boldsymbol{D}}_s)^{-1}\mathrm{diag}(\boldsymbol{P}_s)\right)\boldsymbol{q}_{T-1}\right\} \geq 0 \quad \text{a.s. (under } h_D\text{)},$$

$$\lim_{s\to\infty}\mathrm{E}_t^h[\beta^{s-t}u'(C_s)(P_{0,s}q_{0,s} + \boldsymbol{P}_s^\intercal\boldsymbol{q}_s)] = 0;$$

with $u(C_s) = C_s^{1-\gamma}/(1-\gamma)$, and where

$$h_D(\hat{\boldsymbol{D}}_{s+1}|\hat{\boldsymbol{D}}_s) \text{ solves } \mathcal{P}_{\mathcal{I}} \text{ given } d(\boldsymbol{D}_s, \hat{\boldsymbol{D}}_s) \text{ and } \kappa, \qquad \forall s \geq t,$$
$$g_D(\boldsymbol{D}_{s+1}|\boldsymbol{D}_s) \text{ is given}, \qquad \forall s \geq t.$$

The crucial difference from before is that in the feasible formulation of the consumption and portfolio choice problem the expectation is now taken with respect to the endogenous subjective probability density function for stochastic fruit-dividends, which itself has to be obtained as an optimal solution to the auxiliary informational problem.

The corresponding Bellman equation becomes:

$$v(\{q_{0,t-1}, \boldsymbol{q}_{t-1}\}, \hat{\boldsymbol{D}}_t) = \max_{C_t,\{q_{0,t},\boldsymbol{q}_t\}}\left\{u(C_t) + \beta\mathrm{E}_t^h\left[v(\{q_{0,t}, \boldsymbol{q}_t\}, \hat{\boldsymbol{D}}_{t+1})\right]\right\}$$

subject to

$$C_t + P_{0,t}q_{0,t} + \boldsymbol{P}_t^\intercal\boldsymbol{q}_t = q_{0,t-1} + (\boldsymbol{P}_t + \hat{\boldsymbol{D}}_t)^\intercal\boldsymbol{q}_{t-1},$$

domain restriction $C_t, \{q_{0,t}, \boldsymbol{q}_t\} \in \mathbb{R}_+ \times \mathbb{R}^{K+1}$, as well as the same no-Ponzi-schemes condition; also with the same utility function specification, and where

$$h_D(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t) := \int_{\mathrm{supp}(g_D)} f(\boldsymbol{D}_{t+1}, \hat{\boldsymbol{D}}_{t+1}|\boldsymbol{D}_t, \hat{\boldsymbol{D}}_t)\,d\boldsymbol{D}_{t+1},$$

$$f_D(\boldsymbol{D}_{t+1}, \hat{\boldsymbol{D}}_{t+1}|\boldsymbol{D}_t, \hat{\boldsymbol{D}}_t) := \arg\left\{\min_{f(\cdot,\cdot)}\mathrm{E}^f\left[d(v^\sharp(\{q_{0,t}, \boldsymbol{q}_t\}, \boldsymbol{D}_{t+1}), v(\{q_{0,t}, \boldsymbol{q}_t\}, \hat{\boldsymbol{D}}_{t+1}))\right]\right.$$
$$\left.\text{s.t. } \mathcal{I}(g_D(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t); h_D(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t)) \leq \kappa\right\},$$

$$g_D(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t) \text{ is given}.$$

The Bellman equation's formulation is standard except that the probability density function $h_D(\cdot)$ with respect to which it is defined stems from the solution to auxiliary sub-problem $\mathcal{P}_{\mathcal{I}}$.[1]

B Solution: Technical details

This Apendix presents the solution to feasible consumption and portfolio choice problem $\mathcal{P}_{\mathcal{QI}}$. We start with the consumption and investment segment of the larger problem, dealing with the informational sub-problem afterwards. Clearly segregated formulations of these two sub-problems allow to formally solve each of them pretty much independently.

B.1 Solution to consumption and investment sub-problem

Here we solve problem $\mathcal{P}_{\mathcal{QI}}$ taking $h_D(\hat{\boldsymbol{D}}_{t+1})$ as given, i.e. focusing only on expressions (PQI-1)–(PQI-2) while respecting the domain, no-Ponzi-schemes and utility function restrictions. Essentially, this is a portfolio choice problem of Samuelson (1969), as well as Merton (1969), with price behavior related to underlying dividend dynamics as in Lucas (1978), and Breeden (1979).

First-order necessary conditions for the interior optimum is of the usual form:

$$P_{0,t} = \mathrm{E}_t^h\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}\right] = \mathrm{E}_t^h\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right], \tag{B.1}$$

$$\boldsymbol{P}_t = \mathrm{E}_t^h\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}\left(\boldsymbol{P}_{t+1}+\hat{\boldsymbol{D}}_{t+1}\right)\right] = \mathrm{E}_t^h\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\boldsymbol{P}_{t+1}+\hat{\boldsymbol{D}}_{t+1}\right)\right]. \tag{B.2}$$

We do not provide the full argument, and only mention the importance of realizing that due to i.i.d.-assumption, $\mathrm{E}_t^h\left[\beta\left(\widehat{\boldsymbol{q}}^\intercal\hat{\boldsymbol{D}}_{t+1}\right)^{-\gamma}\left(\boldsymbol{P}(\hat{\boldsymbol{D}}_{t+1})+\hat{\boldsymbol{D}}_{t+1}\right)\right]$ ends up being just a vector of constants. Leaving verification to the reader, we simply state that the optimal solution to consumption and investment sub-parts of the full problem is characterized by the expressions below (unfortunately, completely closed-form analytical solutions are not available in general even for the unconstrained problem $\mathcal{P}_{\mathcal{Q}}$):

$$C_t^* = (1-\beta)W_t = (\boldsymbol{q}_{t-1})^\intercal\boldsymbol{D}_t, \tag{B.3}$$

$$P_{0,t}^*q_{0,t}^* + (\boldsymbol{P}_t^*)^\intercal\boldsymbol{q}_t^* = \beta W_t, \tag{B.4}$$

$$\{q_{0,t}^*, \boldsymbol{q}_t^*\} = \{0, \widehat{\boldsymbol{q}}\}, \tag{B.5}$$

$$P_{0,t}^* = P_0(\boldsymbol{D}_t) = \beta(\widehat{\boldsymbol{q}}^\intercal\boldsymbol{D}_t)^\gamma\,\mathrm{E}_t^h\left[\frac{1}{(\widehat{\boldsymbol{q}}^\intercal\hat{\boldsymbol{D}}_{t+1})^\gamma}\right], \tag{B.6}$$

$$\boldsymbol{P}_t^* = \boldsymbol{P}(\boldsymbol{D}_t) = \frac{\beta}{1-\beta}(\widehat{\boldsymbol{q}}^\intercal\boldsymbol{D}_t)^\gamma\,\mathrm{E}_t^h\left[\frac{1}{(\widehat{\boldsymbol{q}}^\intercal\hat{\boldsymbol{D}}_{t+1})^\gamma}\hat{\boldsymbol{D}}_{t+1}\right], \tag{B.7}$$

$$v_t^* = v(\{q_{0,t-1}, \boldsymbol{q}_{t-1}\}, \boldsymbol{D}_t) = AW_t^{1-\gamma}, \tag{B.8}$$

[1] Note that the no-Ponzi-schemes constraint here holds also for the original probability distribution $g_D(\cdot)$ as long as original and simplified distributions are absolutely continuous with respect to each other.

where, according to the definition in (4),

$$W_t = q_{0,t-1} + (\boldsymbol{P}_t^* + \boldsymbol{D}_t)^\intercal \boldsymbol{q}_{t-1},$$

and, in line with (7),

$$A = \frac{(1-\beta)^{-\gamma}}{1-\gamma}.$$

In the optimum, consumption and total investments are each constant shares of current wealth; and the value function takes the same CRRA form as the utility function, only in terms of wealth.

B.2 Updating of the mean

Notice that

$$\begin{aligned}\hat{\boldsymbol{\mu}}_r(\widehat{\omega}_t) &= \boldsymbol{\mu}_r + \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t) = \boldsymbol{\mu}_r + \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\mathbf{1}(1-\widehat{\omega}_t) \neq \\ &\neq \boldsymbol{\mu}_r + \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t = \boldsymbol{\mu}_r + \check{\boldsymbol{\mu}}_r = \\ &= \hat{\boldsymbol{\mu}}_r\end{aligned}$$

in general. (For $\boldsymbol{\omega}_t$ with positive elements, in the interior solution case it is easy to see that $\hat{\boldsymbol{\mu}}_r(\widehat{\omega}_t) < \hat{\boldsymbol{\mu}}_r$; but this does not always hold in the boundary solution case, as can be shown by a simple counterexample.) However, Proposition 1.1 states that optimal accounting for the discrepancy between original, $\boldsymbol{\Sigma}_r$, and simplified, $\hat{\boldsymbol{\Sigma}}_r$, variance-covariance matrices requires using the latter value for the mean, $\hat{\boldsymbol{\mu}}_r$.

Less formally, we may posit that the mean $\hat{\boldsymbol{\mu}}_r(\widehat{\omega}_t)$ is trained over time off achieved decision outcomes, thus approaching $\hat{\boldsymbol{\mu}}_r$ in the course of "supervised learning".

Alternatively and more formally, we may postulate the following procedure for iterative updating of the mean. In each iteration ι of the optimization process, proposed choice of parameter value $\boldsymbol{\theta}_\iota := \{q_{0,t,\iota}, \boldsymbol{q}_{t,\iota}\}$ that has been accepted is immediately reflected in the corresponding value of $\boldsymbol{\omega}_{t,\iota}$ (which is possible since the latter is then just a function of known values of $\{P_{0,t}, \boldsymbol{P}_t\}$, W_t as well as $\{q_{0,t,\iota}, \boldsymbol{q}_{t,\iota}\}$; and with such auxiliary routine embedded into function $\varphi(\boldsymbol{x}|\boldsymbol{\theta}_\iota)$). In turn, this update allows to compute the values of $\check{\boldsymbol{\mu}}_{r,\iota}$ and $\hat{\boldsymbol{\mu}}_{r,\iota}$. Remember that conditional on the value of $\check{\boldsymbol{\mu}}_{r,\iota}$, distortion function from Proposition 1.3 is otherwise invariant, hence the rest of the solution to informational problem is unaffected, and results of Theorems 1–2 still hold except for updated values of $\check{\boldsymbol{\mu}}_{r,\iota}$ and $\hat{\boldsymbol{\mu}}_{r,\iota}$. Since our environment is sufficiently "well-behaved", both $\boldsymbol{\theta}_\iota$ and $\hat{\boldsymbol{\mu}}_{r,\iota}$ will converge to their optimal values $\boldsymbol{\theta}^*$ and $\hat{\boldsymbol{\mu}}_r$ simultaneously. As a result, we have the following Proposition.

Proposition B.1 (Specific Solution to Informational Problem: Representation in Economic Terms with Updating of the Mean)**.** *Assume the procedure for iterative updating of the mean described in the text above. Then statement of Theorem 2 holds for* $\check{\boldsymbol{\mu}}_r(\widehat{\omega}_t)$ *and* $\hat{\boldsymbol{\mu}}_r(\widehat{\omega}_t)$ *replaced with, respectively,* $\check{\boldsymbol{\mu}}_r$ *and* $\hat{\boldsymbol{\mu}}_r$ *throughout.*

Proof. See Appendix §D.5. ∎

Note that the result of Proposition B.1 is achieved for any admissible starting value of $\widehat{\omega}_t \in \mathbb{R}_+$. It is also noteworthy that postulated iterative updating procedure is akin to the (iterative or continuous) updating requirement discussed in Appendix §J, which we have ruled out appealing to robustness in Proposition 1.2 instead; but at this point the problem is much simpler and requirements needed for implementing the procedure seem more realistic.

B.3 Solution to informational sub-problem

Now we turn to the informational part of the larger problem $\mathcal{P}_{\mathcal{QI}}$. It is basically solved in the main text and crucial details of the solution are presented in §3.3 (with §B.2 furnishing some auxiliary results). The rest is available here.

Turning to informational coherence, optimal solution to the informational sub-problem amounts to the following joint probability density:

$$
\begin{aligned}
f(\boldsymbol{x},\hat{\boldsymbol{x}}) &= f(\boldsymbol{x}|\hat{\boldsymbol{x}})h(\hat{\boldsymbol{x}}) = \\
&= (2\pi)^{-\frac{K}{2}}\left|\boldsymbol{\Psi}\right|^{-\frac{1}{2}} e^{-\frac{1}{2}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}})^{\intercal}\boldsymbol{\Psi}^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}})} \times (2\pi)^{-\frac{K}{2}}\left|\hat{\boldsymbol{\Sigma}}\right|^{-\frac{1}{2}} e^{-\frac{1}{2}(\hat{\boldsymbol{x}}-\hat{\boldsymbol{\mu}})^{\intercal}\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{x}}-\hat{\boldsymbol{\mu}})} = \\
&= (2\pi)^{-\frac{2K}{2}}\left|\begin{bmatrix}\boldsymbol{\Sigma} & \hat{\boldsymbol{\Sigma}}\\ \hat{\boldsymbol{\Sigma}} & \hat{\boldsymbol{\Sigma}}\end{bmatrix}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\begin{bmatrix}\boldsymbol{x}-\boldsymbol{\mu}\\ \hat{\boldsymbol{x}}-\hat{\boldsymbol{\mu}}\end{bmatrix}^{\intercal}\begin{bmatrix}\boldsymbol{\Sigma} & \hat{\boldsymbol{\Sigma}}\\ \hat{\boldsymbol{\Sigma}} & \hat{\boldsymbol{\Sigma}}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{x}-\boldsymbol{\mu}\\ \hat{\boldsymbol{x}}-\hat{\boldsymbol{\mu}}\end{bmatrix}\right),
\end{aligned}
$$

which, after substituting $\boldsymbol{\Xi}^{\intercal}\boldsymbol{r}_{t+1}$ for $\boldsymbol{x}$, $\boldsymbol{\Xi}^{\intercal}\hat{\boldsymbol{r}}_{t+1}$ for $\hat{\boldsymbol{x}}$, $\boldsymbol{\Xi}^{\intercal}\boldsymbol{\mu}_r$ for $\boldsymbol{\mu}$, $\boldsymbol{\Xi}^{\intercal}\hat{\boldsymbol{\mu}}_r$ for $\hat{\boldsymbol{\mu}}$, $\boldsymbol{\Xi}^{\intercal}\boldsymbol{\Sigma}_r\boldsymbol{\Xi}$ for $\boldsymbol{\Sigma}$, and $\boldsymbol{\Xi}^{\intercal}\hat{\boldsymbol{\Sigma}}_r\boldsymbol{\Xi}$ for $\hat{\boldsymbol{\Sigma}}$, produces $f_r(\boldsymbol{r}_{t+1},\hat{\boldsymbol{r}}_{t+1})$:

$$
\begin{aligned}
f(\boldsymbol{x},\hat{\boldsymbol{x}}) &= (2\pi)^{-\frac{2K}{2}}\left|\begin{bmatrix}\boldsymbol{\Sigma}_r & \hat{\boldsymbol{\Sigma}}_r\\ \hat{\boldsymbol{\Sigma}}_r & \hat{\boldsymbol{\Sigma}}_r\end{bmatrix}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\begin{bmatrix}\boldsymbol{r}_{t+1}-\boldsymbol{\mu}_r\\ \hat{\boldsymbol{r}}_{t+1}-\hat{\boldsymbol{\mu}}_r\end{bmatrix}^{\intercal}\begin{bmatrix}\boldsymbol{\Sigma}_r & \hat{\boldsymbol{\Sigma}}_r\\ \hat{\boldsymbol{\Sigma}}_r & \hat{\boldsymbol{\Sigma}}_r\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{r}_{t+1}-\boldsymbol{\mu}_r\\ \hat{\boldsymbol{r}}_{t+1}-\hat{\boldsymbol{\mu}}_r\end{bmatrix}\right) = \\
&=: f_r(\boldsymbol{r}_{t+1},\hat{\boldsymbol{r}}_{t+1}).
\end{aligned}
$$

That is, $f(\cdot,\cdot)$ and $f_r(\cdot,\cdot)$ have the same multivariate-$\mathcal{N}$ form, i.e. for parameters consisting of mean vector $\boldsymbol{\Theta}_1$ and variance-covariance matrix $\boldsymbol{\Theta}_2$,

$$
f(\boldsymbol{\chi},\hat{\boldsymbol{\chi}}|\boldsymbol{\Theta}_1,\boldsymbol{\Theta}_2) = f_r(\boldsymbol{\chi},\hat{\boldsymbol{\chi}}|\boldsymbol{\Theta}_1,\boldsymbol{\Theta}_2) \text{ is } \mathcal{N}(\boldsymbol{\Theta}_1,\boldsymbol{\Theta}_2) \quad \forall\boldsymbol{\chi},\hat{\boldsymbol{\chi}}\in\mathbb{R}^K.
$$

Obviously, analogous relationship holds for $g(\cdot)$ and $g_r(\cdot)$, as well as $h(\cdot)$ and $h_r(\cdot)$.

B.4 Merging two sub-problems' solutions

Finally, we use the results from §B.1 and §B.3 to tie up the loose ends concerning the probability distributions of dividends.

Having got the solution for $\boldsymbol{P}_t^*$ in (B.7), we can combine definition (5) with the result of Theorem 2 to "reverse-engineer" approximating probability density function for dividends so that by construction it would be coherent with approximating density of returns deduced in the Theorem:

$$h_r(\hat{\boldsymbol{r}}_{t+1}) = (2\pi)^{-\frac{K}{2}} |\hat{\boldsymbol{\Sigma}}_r|^{-\frac{1}{2}} e^{-\frac{1}{2}(\hat{\boldsymbol{r}}(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t)-\hat{\boldsymbol{\mu}}_r)^\intercal \hat{\boldsymbol{\Sigma}}_r^{-1}(\hat{\boldsymbol{r}}(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t)-\hat{\boldsymbol{\mu}}_r)} =: h_D(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t),$$

where

$$\begin{aligned}\hat{\boldsymbol{r}}(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t) = \ln \hat{\boldsymbol{R}}(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t) = \ln\left(\operatorname{diag}\left(\boldsymbol{P}(\boldsymbol{D}_t)\right)^{-1}\left(\boldsymbol{P}(\hat{\boldsymbol{D}}_{t+1}) + \hat{\boldsymbol{D}}_{t+1}\right)\right) :=\\ := \ln\left(\operatorname{diag}\left(\frac{\beta}{1-\beta}(\widehat{\boldsymbol{q}}^\intercal \boldsymbol{D}_t)^\gamma \boldsymbol{E}\right)^{-1}\left(\frac{\beta}{1-\beta}(\widehat{\boldsymbol{q}}^\intercal \hat{\boldsymbol{D}}_{t+1})^\gamma \boldsymbol{E} + \hat{\boldsymbol{D}}_{t+1}\right)\right).\end{aligned}$$

Here, we have introduced for the following constant vector a shortcut notation

$$\boldsymbol{E} := \mathrm{E}_t^h\left[\frac{1}{(\widehat{\boldsymbol{q}}^\intercal \hat{\boldsymbol{D}}_{t+1})^\gamma}\hat{\boldsymbol{D}}_{t+1}\right].$$

Since in equilibirum prices are dictated by the solution to consumption and investment subproblem from §B.1, we use the same optimal price function (B.7), together with definition (5) and assumption (8), also to back out the true density for dividends:

$$g_r(\boldsymbol{r}_{t+1}) = (2\pi)^{-\frac{K}{2}} |\boldsymbol{\Sigma}_r|^{-\frac{1}{2}} e^{-\frac{1}{2}(\boldsymbol{r}(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t)-\boldsymbol{\mu}_r)^\intercal \boldsymbol{\Sigma}_r^{-1}(\boldsymbol{r}(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t)-\boldsymbol{\mu}_r)} =: g_D(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t),$$

where

$$\begin{aligned}\boldsymbol{r}(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t) = \ln \boldsymbol{R}(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t) = \ln\left(\operatorname{diag}\left(\boldsymbol{P}(\boldsymbol{D}_t)\right)^{-1}\left(\boldsymbol{P}(\boldsymbol{D}_{t+1}) + \boldsymbol{D}_{t+1}\right)\right) :=\\ := \ln\left(\operatorname{diag}\left(\frac{\beta}{1-\beta}(\widehat{\boldsymbol{q}}^\intercal \boldsymbol{D}_t)^\gamma \boldsymbol{E}\right)^{-1}\left(\frac{\beta}{1-\beta}(\widehat{\boldsymbol{q}}^\intercal \boldsymbol{D}_{t+1})^\gamma \boldsymbol{E} + \boldsymbol{D}_{t+1}\right)\right).\end{aligned}$$

Note that constant vector $\boldsymbol{E}$ used here in the determination of true $\boldsymbol{r}(\boldsymbol{D}_{t+1}|\boldsymbol{D}_t)$ is the same as in the case of approximating $\hat{\boldsymbol{r}}(\hat{\boldsymbol{D}}_{t+1}|\hat{\boldsymbol{D}}_t)$, and is still defined in terms of simplified probability density according to (B.4).

The corresponding result for $f_r(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1})$ and $f_D(\boldsymbol{D}_{t+1}, \hat{\boldsymbol{D}}_{t+1}|\boldsymbol{D}_t, \hat{\boldsymbol{D}}_t)$ follows in a similar manner, thus validating informational coherence.

C Related literature

Related themes in the literature only briefly mentioned in the main text include:

- information-theoretic methods: classical works (Shannon, 1948; Jaynes, 2003; Cover and Thomas, 2006; MacKay, 2003; in economics Marschak, 1959, 1968, 1971), rational inattention based on mutual information (Sims, 1998, 2003, 2006, 2010; Matějka and Sims, 2010; Matějka and McKay, 2014; Matějka, 2014a, 2014b; Caplin and Dean, forthcoming, 2013; Ravid, 2016; also refer to Woodford, 2012, 2014), model diagnostics and belief measurement based on relative entropy (Stutzer, 1995, 1996; Hansen and Sargent, 2007a, 2007b; Hansen, 2007; Backus et al., 2014; Ghosh et al., 2013; Chen et al., 2015; Alvarez and Jermann, 2005; overview available in Hansen, 2014);

- statistical learning and simplification: "coarsening" (Al-Najjar and Pai, 2014), regularization (Chen and Haykin, 2002; Bickel and Li, 2006), shrinkage (Stein, 1956; James and Stein, 1961; Tibshirani, 1996; in finance Ledoit and Wolf, 2003, 2004a, 2004b; Jagannathan and Ma, 2003; Won et al., 2012; Jorion, 1985; 1986);

- bounded rationality focusing on decision costs and ensuing simplification: classical works (Simon 1997, 1957; Gigerenzer and Selten, 2001; Gigerenzer et al., 2000; Gilovich et al., 2002), modern works (Gabaix and Laibson, 2000, 2005; Gabaix et al., 2006; Fuster et al., 2012; Bordalo et al., 2016; Gabaix, 2014a, 2014b; Mullainathan, 2002b; Jehiel, 2005; also see Carroll, 2003), with costs due to memory limitations (Gilboa and Schmeidler, 1995; Wilson, 2014; Mullainathan, 2002a);

- bounded rationality focusing on decision criterion and ensuing belief distortions, including "optimism"/"pessimism" and "overconfidence"/"doubt": endogenous distortions (Hansen, 2007; Hansen and Sargent 2007a, 2007b; Brunnermeier and Parker, 2005; Brunnermeier et al., 2007; Brunnermeier et al., 2013); exogenous distortions (Cecchetti et al., 2000; Abel, 2002; Scheinkman and Xiong, 2003; Peng and Xiong, 2006);

- experimental evidence: entropy reduction (Gabaix and Laibson, 2000; Goldstone and Hendrickson, 2010; Fleming et al., 2013), overconfidence (Camerer, 1995);

- neuroscience and psychology: textbooks (Dayan and Abbott, 2001; Squire et al., 2008), thematic volumes (Baddeley et al., 2000; Doya et al., 2007; Glimcher et al., 2008), journal review issues (Schultz, 2008; Bayer, 2008; Bunge, 2005; Pammi and Srinivasan, 2013), information-processing capacity (classical papers Miller, 1956; Barlow, 1961), working memory capacity (Shiffrin and Nosofsky, 1994; Kleinberg and Kaufman, 1971; Bor et al., 2003; Owen, 2004; Bor and Owen, 2006; Migliore et al., 2008; with review given in Cowan, 2000), selected models (Eliasmith et al., 2012; Rao, 2010; as well as Padoa-Schioppa and Rustichini, 2014, 2015);

- asset pricing and portfolio holding regularities: "asset classes" ("asset allocation" in Brinson et al., 1986, 1991; Sharpe, 1992; Fama and French, 1993; Doeswijk et al., 2014; with "strategic" undertone and stocks versus bonds in Brennan et al., 1997; Campbell and Viceira, 2002b; Wachter, 2010; Shiller and Beltratti, 1992; Canner et al., 1997; Campbell et al., 2003; Campbell and Ammer, 1993; Fleming et al., 1998; Connolly et al., 2005; Andersson et al., 2008; Guidolin and Timmermann, 2007; Li, 2002, 2003; Yang et al., 2009; "comovement" in Shiller, 1989; Pindyck and Rotemberg, 1990; Kumar and Lee, 2006; Barber et al., 2009; Barberis et al., 2005; Veldkamp, 2006; "style investing" in Barberis and Shleifer, 2003; Peng and Xiong, 2006; Brown and Goetzmann, 1997; Froot and Teo, 2008; Kumar, 2009; Teo and Woo, 2004; Chan et al., 2000; Wahal and Yavuz, 2013; Cooper et al., 2005; Boyer, 2011; Froot and Teo, 2008; Choi and Sias, 2009; Jame and Tong, 2014), "underdiversification puzzle" (Blume and Friend, 1975; Statman, 1987; Kelly, 1995; French and Poterba, 1991; with analysis and explanations offered in Polkovnichenko, 2005; Goetzmann and Kumar, 2008; Anderson, 2013; Goetzmann et al., 2005; Li, 2003; Van Nieuwerburgh and Veldkamp, 2009, 2010);

- "free energy" and decision-making (Ortega and Braun, 2013; Friston, 2009, 2010; Still et al., 2012).

D Proofs

D.1 Proof of Lemma 1

Proof. In information theory $\mathcal{P}_{\mathcal{I}}$ is known as the problem of finding the distortion rate function (e.g., see Cover and Thomas 2006). This is a standard calculus of variations problem that can be solved using the method of Lagrange multipliers.

Form the Lagrangian functional:

$$\begin{aligned}
\mathcal{L} := & \int_{\text{supp}(h)} \int_{\text{supp}(g)} d(\boldsymbol{x}, \hat{\boldsymbol{x}})\, f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\boldsymbol{x}\, d\hat{\boldsymbol{x}} + \\
& + \lambda \Bigg[\int_{\text{supp}(h)} \int_{\text{supp}(g)} f(\boldsymbol{x}, \hat{\boldsymbol{x}}) \ln f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\boldsymbol{x}\, d\hat{\boldsymbol{x}} - \\
& \quad - \int_{\text{supp}(h)} \underbrace{\left(\int_{\text{supp}(g)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\boldsymbol{x} \right)}_{h(\hat{\boldsymbol{x}})} \ln \underbrace{\left(\int_{\text{supp}(g)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\boldsymbol{x} \right)}_{h(\hat{\boldsymbol{x}})} d\hat{\boldsymbol{x}} - \\
& \quad - \int_{\text{supp}(g)} g(\boldsymbol{x}) \ln g(\boldsymbol{x})\, d\boldsymbol{x} - \kappa \Bigg] + \\
& + \mu(\boldsymbol{x}) \left[\int_{\text{supp}(h)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\hat{\boldsymbol{x}} - g(\boldsymbol{x}) \right] + \\
& + \nu(\boldsymbol{x}, \hat{\boldsymbol{x}}) \left[-f(\boldsymbol{x}, \hat{\boldsymbol{x}}) \right].
\end{aligned}$$

The integrability condition ensures the optimum exists. Absolute continuity rules out boundary solutions. The objective is convex in $f(\hat{\boldsymbol{x}}|\boldsymbol{x})$ for fixed $g(\boldsymbol{x})$, so the first order condition with respect to $f(\boldsymbol{x}, \hat{\boldsymbol{x}})$ is sufficient for interior minimum. Equalize the corresponding functional derivative to 0 to obtain

$$\begin{aligned}
0 =: \frac{\delta \mathcal{L}}{\delta f(\boldsymbol{x}, \hat{\boldsymbol{x}})} = & \, d(\boldsymbol{x}, \hat{\boldsymbol{x}}) + \lambda \left\{ \ln f(\boldsymbol{x}, \hat{\boldsymbol{x}}) + 1 - \ln \left(\int_{\text{supp}(g)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\boldsymbol{x} \right) - 1 \right\} + \\
& + \mu(\boldsymbol{x}) - \nu(\boldsymbol{x}, \hat{\boldsymbol{x}}).
\end{aligned}$$

Rearranging yields the following solution to minimization problem:

$$f(\boldsymbol{x}|\hat{\boldsymbol{x}}) = e^{\frac{1}{\lambda}\nu(\boldsymbol{x},\hat{\boldsymbol{x}}) - \frac{1}{\lambda}\mu(\boldsymbol{x}) - \frac{1}{\lambda}d(\boldsymbol{x},\hat{\boldsymbol{x}})}.$$

□

D.2 Proof of Proposition 1, with additional comments

The statement and its proof is split into three auxiliary propositions.

Proposition 1.1 (Approximate Distortion Function). *Let distortion function*

$$d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) = (\ln W_{t+1} - \ln \hat{W}_{t+1})^2$$

be a random scalar defined in the context of problem $\mathcal{P}_{\mathcal{QI}}$.

Then, under distributional assumptions given, it approximately equals

$$d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) \approx \left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)\right)^2,$$

where

$$\boldsymbol{\omega}_t := \operatorname{diag}(\boldsymbol{P}_t)\boldsymbol{q}_t \,/\, W_t$$

is a K-vector of the shares of wealth invested in risky assets; $\hat{\boldsymbol{\Sigma}}_r$ is a variance-covariance matrix for $\hat{\boldsymbol{r}}_{t+1}$; while the mean of simplified random variable $\hat{\boldsymbol{\mu}}_r$ equals

$$\hat{\boldsymbol{\mu}}_r := \boldsymbol{\mu}_r + \check{\boldsymbol{\mu}}_r,$$

with bias $\check{\boldsymbol{\mu}}_r$ defined as

$$\check{\boldsymbol{\mu}}_r := \frac{1}{2}\operatorname{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t.$$

As interval between time periods t and $(t+1)$ shrinks, in the limit the approximation turns into exact expression.

Proof. See Appendix §D.2.1.

□

The approximation is based on continuous-time representation of the stochastic processes involved ($\ln W_{t+1}$ and $\boldsymbol{\omega}_t^\intercal \boldsymbol{r}_{t+1}$), it is relatively innocuous. The general approach is related to that of Campbell and Viceira (2002a). On the other hand, similar approximation result can also be obtained directly using second-order Taylor expansion.

Approximate distortion function from Proposition 1.1 is purpose-specific: it adjusts to and depends on $\boldsymbol{\omega}_t$, the shares of wealth considered for being invested in risky trees (note that riskless tree's share does not enter the expression, thus even the scales of three summands are not pinned down here). This introduces a sort of endogeneity, circularity to the choice of what shares to use: a contemplated portfolio allocation generates respective shares, thus inducing the distortion function and the corresponding aproximating distribution, which in turn will lead to another candidate portfolio choice and updated set of shares that differ from those used for the latest iteration of an approximating distribution.

To resolve this dilemma, we motivate the choice of wealth shares to use by the minimization of maximum loss, robustness to "worst-case" scenarios (cf. Hansen and Sargent, 2007). This approach will allow us to get rid of the connection to actual portfolio allocations or net supplies altogether. Results of its application are condensed in the following Proposition. (Appendix §J discusses alternative resolutions of this circularity dilemma.)

Proposition 1.2 (Robust Strategy)**.** *Let player Agent solve problem $\mathcal{P}_{\mathcal{I}}$ with disortion function defined as in the statement of Proposition 1.1:*

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f[d(\boldsymbol{r}_{t+1},\hat{\boldsymbol{r}}_{t+1})] \iff \min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f\left[(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1}-\hat{\boldsymbol{r}}_{t+1}+\check{\boldsymbol{\mu}}_r))^2\right]$$

subject to standard information constraint and technical restrictions (i.e., λ-, $\mu(\boldsymbol{x})$- and $\nu(\boldsymbol{x},\hat{\boldsymbol{x}})$-constraints). Let player Nature maximize the same objective function with respect to $\boldsymbol{\omega}_{1K,t}$, that is solve the following problem:

$$\max_{\boldsymbol{\omega}_t} \mathrm{E}^f[d(\boldsymbol{r}_{t+1},\hat{\boldsymbol{r}}_{t+1})] \iff \max_{\boldsymbol{\omega}_t} \mathrm{E}^f\left[(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1}-\hat{\boldsymbol{r}}_{t+1}+\check{\boldsymbol{\mu}}_r))^2\right]$$

subject to

$$\begin{aligned} \boldsymbol{\omega}_t^\intercal\mathbf{1} &\leq \widehat{\omega}_t^N && \forall\widehat{\omega}_t^N \in \mathbb{R}_+,\\ \omega_{k,t} &\geq 0 && \forall k \in \{1,\ldots,K\}. \end{aligned}$$

Then a simultaneous-move game posesses a Nash-equilibrium characterized by the following strategies ($\forall\widehat{\omega}_t^A,\widehat{\omega}_t^N \in \mathbb{R}_+$): Agent plays

$$f_r^\star(\boldsymbol{r}_{t+1},\hat{\boldsymbol{r}}_{t+1}) := \arg\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f\left[\sum_k \frac{1}{K}\left(\widehat{\omega}_t^A\right)^2\left(r_{k,t+1}-\hat{r}_{k,t+1}+\check{\mu}_{r,k}(\widehat{\omega}_t^A)\right)^2\right],$$

where

$$\check{\mu}_{r,k}(\widehat{\omega}_t^A) := \frac{1}{2}(\sigma_{r,k}^2-\hat{\sigma}_{r,k}^2)(1-\widehat{\omega}_t^A) \qquad \forall k \in \{1,\ldots,K\};$$

Nature plays a mixed strategy

$$\begin{aligned} \omega_{k,t} &:= \widehat{\omega}_t^N, && k = k^\star,\\ \omega_{k,t} &:= 0, && k \neq k^\star, \end{aligned}$$

where

$$k^\star \sim \mathcal{U}(\{1,\ldots,K\}).$$

A game where player Nature is a second-mover posesses a Nash-equilibrium characterized by the same strategies.

Proof. See Appendix §D.2.2. □

Fixed scalars $\widehat{\omega}_t^A$ and $\widehat{\omega}_t^N$ in the Proposition above serve the role of unknown constant parameters denoting some chosen value of total share of wealth invested in risky assets in Agent's and Nature's problems, respectively. For instance, $\widehat{\omega}_t^A$ may be thought as implicit borrowing/collateral constraint.

Also note that constraint $\omega_{k,t} \geq 0 \; \forall k \in \{1, \ldots, K\}$ in Nature's problem ensures consistency with general equilibrium.

Proposition 1.2 is of interest to us not in itself, but as a means of further refining of the distoriton function to be used. Such a distortion function is formulated in Proposition 1.3.

Proposition 1.3 (Robust Approximate Distortion Function)**.** *Robust strategy given by the statement of Proposition 1.2, and assuming total share of wealth invested in risky assets agrees with the definition in Proposition 1.1, i.e.*

$$\widehat{\omega}_t := \mathbf{1}^\intercal \boldsymbol{\omega}_t = \mathbf{1}^\intercal \mathrm{diag}(\boldsymbol{P}_t) \boldsymbol{q}_t \, / \, W_t,$$

induces the following distortion function:

$$d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) := (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t))^\intercal (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t)),$$

where

$$\begin{aligned} \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t) &:= \frac{1}{2} \mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)(1 - \widehat{\omega}_t) = \\ &= \frac{1}{2} \mathrm{diag}(\sigma_{r,1}^2 - \hat{\sigma}_{r,1}^2, \cdots, \sigma_{r,K}^2 - \hat{\sigma}_{r,K}^2) \mathbf{1} (1 - \widehat{\omega}_t). \end{aligned}$$

Proof. See Appendix §D.2.3. □

D.2.1 Proof of Proposition 1.1

Proof. Consider continuous-time dynamics of dividend and price processes from problem $\mathcal{P}_{\mathcal{Q}}$ (their counterparts from problem $\mathcal{P}_{\mathcal{QI}}$ are analogous to ones below):

$$\begin{aligned} d\boldsymbol{D}_t &:= \mathrm{diag}(\boldsymbol{D}_t) \boldsymbol{\mu}_D dt + \mathrm{diag}(\boldsymbol{D}_t) \boldsymbol{\sigma}_D d\boldsymbol{B}_t, \\ dP_{0,t} &:= r_{0,t} P_{0,t} dt, \\ d\boldsymbol{P}_t &:= \mathrm{diag}(\boldsymbol{P}_t)(\boldsymbol{\mu}_P + \frac{1}{2} \mathrm{diag}^{-1}(\boldsymbol{\sigma}_P \boldsymbol{\sigma}_P^\intercal)) dt + \mathrm{diag}(\boldsymbol{P}_t) \boldsymbol{\sigma}_P d\boldsymbol{B}_t, \end{aligned}$$

where $\boldsymbol{D}_t$ is the dividend process, $\boldsymbol{\mu}_D$ is a K-sized constant vector, $\boldsymbol{\sigma}_D$ is a $K \times K$ constant matrix, $\boldsymbol{B}_t$ is a standard K-dimensional Brownian motion, $P_{0,t}$ is thought as the money account process with stochastic instantaneous interest rate $r_{0,t}$ (which is equivalent to rolling over just maturing zero coupon bonds), and $\boldsymbol{P}_t$ is the price process corresponding to assets with the dividend process given above.

By Itō's lemma, for $\boldsymbol{P}_t := \boldsymbol{P}(\boldsymbol{D}_t)$ we have:

$$\begin{aligned}
\boldsymbol{\mu}_P &= \operatorname{diag}(\boldsymbol{P}_t)^{-1}\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{D}^\intercal}\operatorname{diag}(\boldsymbol{D}_t)\boldsymbol{\mu}_D + \frac{1}{2}\operatorname{diag}(\boldsymbol{P}_t)^{-1}\boldsymbol{\sigma}_D^\intercal\operatorname{diag}(\boldsymbol{D}_t)\frac{\partial^2 \boldsymbol{P}}{\partial \boldsymbol{D}\partial \boldsymbol{D}^\intercal}\operatorname{diag}(\boldsymbol{D}_t)\boldsymbol{\sigma}_D - \frac{1}{2}\operatorname{diag}^{-1}(\boldsymbol{\sigma}_P\boldsymbol{\sigma}_P^\intercal),\\
\boldsymbol{\sigma}_P &= \operatorname{diag}(\boldsymbol{P}_t)^{-1}\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{D}^\intercal}\operatorname{diag}(\boldsymbol{D}_t)\boldsymbol{\sigma}_D.
\end{aligned}$$

We also have, deducing the (ex-dividend) wealth dynamics from the budget constraint:

$$\begin{aligned}
d\ln \boldsymbol{P}_t &= \boldsymbol{\mu}_P dt + \boldsymbol{\sigma}_P d\boldsymbol{B}_t,\\
dW_t &= W_t\left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{\mu}_P + \frac{1}{2}\operatorname{diag}^{-1}(\boldsymbol{\sigma}_P\boldsymbol{\sigma}_P^\intercal) + \operatorname{diag}(\boldsymbol{P}_t)^{-1}\boldsymbol{D}_t - r_{0,t}\boldsymbol{1}) + r_{0,t} - \frac{C_t}{W_t}\right)dt +\\
&\quad + W_t\boldsymbol{\omega}_t^\intercal\boldsymbol{\sigma}_P d\boldsymbol{B}_t,\\
d\ln W_t &= \left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{\mu}_P + \frac{1}{2}\operatorname{diag}^{-1}(\boldsymbol{\sigma}_P\boldsymbol{\sigma}_P^\intercal) + \operatorname{diag}(\boldsymbol{P}_t)^{-1}\boldsymbol{D}_t - r_{0,t}\boldsymbol{1}) + r_{0,t} - \frac{C_t}{W_t} -\right.\\
&\quad \left. - \frac{1}{2}\boldsymbol{\omega}_t^\intercal\boldsymbol{\sigma}_P\boldsymbol{\sigma}_P^\intercal\boldsymbol{\omega}_t\right)dt + \boldsymbol{\omega}_t^\intercal\boldsymbol{\sigma}_P d\boldsymbol{B}_t,
\end{aligned}$$

where

$$\begin{aligned}
W_t &:= P_{0,t}q_{0,t} + \boldsymbol{P}_t^\intercal\boldsymbol{q}_t,\\
\omega_{0,t} &:= P_{0,t}q_{0,t} \,/\, W_t,\\
\boldsymbol{\omega}_t &:= \frac{1}{W_t}\operatorname{diag}(\boldsymbol{P}_t)\boldsymbol{q}_t.
\end{aligned}$$

Increasing time intervals to $dt = 1$ and setting

$$\begin{aligned}
\boldsymbol{\Sigma}_r &:= \boldsymbol{\sigma}_P\boldsymbol{\sigma}_P^\intercal,\\
\boldsymbol{r}_{t+1} &:= \boldsymbol{\mu}_P + \boldsymbol{\sigma}_P(\boldsymbol{B}_{t+1} - \boldsymbol{B}_t) =: \boldsymbol{\mu}_r + \mathcal{N}(0, \boldsymbol{\Sigma}_r)
\end{aligned}$$

produces a continuous-time approximation to a discrete-time case:

$$\begin{aligned}
\ln W_{t+1} - \ln W_t \approx &\left(\boldsymbol{\omega}_t^\intercal(\operatorname{diag}(\boldsymbol{P}_t)^{-1}\boldsymbol{D}_t - r_{0,t}\boldsymbol{1}) + r_{0,t} - \frac{C_t}{W_t} +\right.\\
&\left. + \frac{1}{2}\boldsymbol{\omega}_t^\intercal\operatorname{diag}^{-1}(\boldsymbol{\Sigma}_r) - \frac{1}{2}\boldsymbol{\omega}_t^\intercal\boldsymbol{\Sigma}_r\boldsymbol{\omega}_t\right) + \boldsymbol{\omega}_t^\intercal\boldsymbol{r}_{t+1},
\end{aligned}$$

with its constrained counterpart being (will verify later in Theorem 2 that log-normality of constrained random variables is admissible, and deduce the value of $\hat{\boldsymbol{\mu}}_r$ shortly)

$$\begin{aligned}
\ln \hat{W}_{t+1} - \ln W_t \approx &\left(\boldsymbol{\omega}_t^\intercal(\operatorname{diag}(\boldsymbol{P}_t)^{-1}\boldsymbol{D}_t - r_{0,t}\boldsymbol{1}) + r_{0,t} - \frac{C_t}{W_t} +\right.\\
&\left. + \frac{1}{2}\boldsymbol{\omega}_t^\intercal\operatorname{diag}^{-1}(\hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}\boldsymbol{\omega}_t^\intercal\hat{\boldsymbol{\Sigma}}_r\boldsymbol{\omega}_t\right) + \boldsymbol{\omega}_t^\intercal\hat{\boldsymbol{r}}_{t+1}.
\end{aligned}$$

Thus,

$$d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) = (\ln W_{t+1} - \ln \hat{W}_{t+1})^2 \approx$$
$$\approx \left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1}) + \frac{1}{2}\boldsymbol{\omega}_t^\intercal \mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}\boldsymbol{\omega}_t^\intercal(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t\right)^2.$$

Since wealth process follows a geometric Brownian motion, equalizing expected growth rates of true $\ln W_{t+1}$ and approximate $\ln \hat{W}_{t+1}$ when volatility $\boldsymbol{\Sigma}_r$ is replaced with $\hat{\boldsymbol{\Sigma}}_r$ necessitates an adjustment to the mean of approximating random variable, $\hat{\boldsymbol{\mu}}_r$. It is easy to see that the correct mean has to be

$$\hat{\boldsymbol{\mu}}_r := \boldsymbol{\mu}_r + \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t =: \boldsymbol{\mu}_r + \check{\boldsymbol{\mu}}_r.$$

Which also is, by the usual mean-as-a-minimum-MSE-estimator logic, the minimizer of

$$\mathrm{E}^f[d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1})] \approx \mathrm{E}^f\left[\left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1}) + \frac{1}{2}\boldsymbol{\omega}_t^\intercal \mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}\boldsymbol{\omega}_t^\intercal(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t\right)^2\right],$$

producing as a result

$$\mathrm{E}^f[d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1})]\Big|_{\hat{\boldsymbol{\mu}}_r = \boldsymbol{\mu}_r + \check{\boldsymbol{\mu}}_r} \approx \boldsymbol{\omega}_t^\intercal \boldsymbol{\Psi}_r \boldsymbol{\omega}_t,$$

where we have introduced (in accordance with Theorem 2) notation

$$\boldsymbol{\Psi}_r := \mathrm{E}^f[(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)^\intercal].$$

Now approximate distortion function can also be formulated as

$$d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) \approx \left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1}) + \frac{1}{2}\boldsymbol{\omega}_t^\intercal \mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}\boldsymbol{\omega}_t^\intercal(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t\right)^2 =$$
$$= \left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)\right)^2.$$

Lastly, keeping time intervals infinitesimally short would leave us in continuous time framework, with the above expressions being exact.

□

D.2.2 Proof of Proposition 1.2

Proof. Consider simultaneous-move game first. The argument proceeds in 4 steps.

1. For a given probability density function $f_r(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1})$, we can write the objective function as

$$\mathrm{E}^f[d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1})] = \mathrm{E}^f\left[\left(\boldsymbol{\omega}_t^\intercal(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)\right)^2\right] = \boldsymbol{\omega}_t^\intercal \boldsymbol{\Psi}_r \boldsymbol{\omega}_t,$$

   where we use (in accordance with Theorem 2) notation

$$\boldsymbol{\Psi}_r := \mathrm{E}^f[(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)^\intercal].$$

2. Nature solves

$$\max_{\boldsymbol{\omega}_t} \mathrm{E}^f[d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1})] \iff \max_{\boldsymbol{\omega}_t} \boldsymbol{\omega}_t^\intercal \boldsymbol{\Psi}_r \boldsymbol{\omega}_t$$

   subject to

$$\begin{aligned} \boldsymbol{\omega}_t^\intercal \mathbf{1} &\leq \widehat{\omega}_t^N && \forall \widehat{\omega}_t^N \in \mathbb{R}_+, \\ \omega_{k,t} &\geq 0 && \forall k \in \{1, \dots, K\}. \end{aligned}$$

   (i) Perfect information case: $\boldsymbol{\Psi}_r$ is known. (Benchmark case.)
   Notice that

$$\begin{aligned} \mathrm{V}\left(\boldsymbol{\omega}_t^\intercal (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r)\right) &= \sum_k \omega_{k,t}^2 \psi_{r,k}^2 + 2 \sum_{k \neq l} \omega_{k,t} \omega_{l,t} \psi_{r,kl} \leq \\ &\leq \sum_k \omega_{k,t}^2 \psi_{r,k}^2 + 2 \sum_{k \neq l} \omega_{k,t} \omega_{l,t} \psi_{r,k} \psi_{r,l} \leq \\ &\leq \left(\widehat{\omega}_t^N\right)^2 \max_{k \in \{1,\dots,K\}} \psi_{r,k}^2. \end{aligned}$$

   Thus, Nature chooses corner solution:

$$\begin{aligned} \omega_{k,t} &:= \widehat{\omega}_t^N, && k = k^\star := \arg \max_{k \in \{1,\dots,K\}} \psi_{r,k}^2 \text{ (randomize if multiplicity)}, \\ \omega_{k,t} &:= 0, && k \neq k^\star. \end{aligned}$$

   (ii) Imperfect information case: $\boldsymbol{\Psi}_r$ unknown. (Simultaneous-move case.)
   Still, Nature would want to choose corner solution. The principle of indifference (principle of insufficient reason) entails uniform distribution for the corner solution's candidate:

$$k^\star \sim \mathcal{U}(\{1, \dots, K\}).$$

   Thus, Nature's move is:

$$\begin{aligned} \omega_{k,t} &:= \widehat{\omega}_t^N, && k = k^\star \sim \mathcal{U}(\{1, \dots, K\}), \\ \omega_{k,t} &:= 0, && k \neq k^\star. \end{aligned}$$

3. Agent anticipates Nature's strategy and formulates the objective function to solve ($\forall \widehat{\omega}_t^A \in \mathbb{R}_+$; notice that $\widehat{\omega}_t^A \neq \widehat{\omega}_t^N$ in general, so coordination between players on

exact share is not necessary for achieving an equilibrium):

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \sum_k \frac{1}{K} \mathrm{E}^f [d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) \mid \omega_{k,t} = \widehat{\omega}_t^A;\, \omega_{l,t} = 0,\, \forall l \neq k] \iff$$

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f \left[\sum_k \frac{1}{K} d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) \middle| \omega_{k,t} = \widehat{\omega}_t^A;\, \omega_{l,t} = 0,\, \forall l \neq k \right] \iff$$

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f \left[\sum_k \frac{1}{K} (\boldsymbol{\omega}_t^\intercal (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r))^2 \middle| \omega_{k,t} = \widehat{\omega}_t^A;\, \omega_{l,t} = 0,\, \forall l \neq k \right] \iff$$

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f \left[\sum_k \frac{1}{K} \left(\boldsymbol{\omega}_t^\intercal (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \frac{1}{2} \mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t) \right)^2 \middle| \omega_{k,t} = \widehat{\omega}_t^A;\, \omega_{l,t} = 0,\, \forall l \right.$$

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f \left[\sum_k \frac{1}{K} \left(\widehat{\omega}_t^A\right)^2 \left(r_{k,t+1} - \hat{r}_{k,t+1} + \frac{1}{2}(\sigma_{r,k}^2 - \hat{\sigma}_{r,k}^2) - \frac{1}{2}(\sigma_{r,k}^2 - \hat{\sigma}_{r,k}^2)\widehat{\omega}_t^A \right)^2 \right] \iff$$

$$\min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f \left[\sum_k \frac{1}{K} \left(\widehat{\omega}_t^A\right)^2 (r_{k,t+1} - \hat{r}_{k,t+1} + \check{\mu}_{r,k}(\widehat{\omega}_t^A))^2 \right]$$

subject to standard informational constraints.

4. In Nash-equilibrium of this simultaneous-move game, players' strategies are as follows ($\forall \widehat{\omega}_t^A, \widehat{\omega}_t^N \in \mathbb{R}_+$):

   - Agent plays

$$f_r^\star(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}) := \arg \min_{f_r(\boldsymbol{r},\hat{\boldsymbol{r}})} \mathrm{E}^f \left[\sum_k \frac{1}{K} \left(\widehat{\omega}_t^A\right)^2 (r_{k,t+1} - \hat{r}_{k,t+1} + \check{\mu}_{r,k}(\widehat{\omega}_t^A))^2 \right];$$

   - Nature plays a mixed strategy

$$\omega_{k,t} := \widehat{\omega}_t^N, \qquad k = k^\star,$$
$$\omega_{k,t} := 0, \qquad k \neq k^\star,$$

   where

$$k^\star \sim \mathcal{U}(\{1, \ldots, K\}).$$

For a game with Nature as a second-mover, the proof requires only a slight modification in step 2, where the perfect information case applies. (Note that step 3 remains unchanged due to previous step's randomization in situation of multiplicity.)

□

D.2.3 Proof of Proposition 1.3

Proof. Plugging $\widehat{\omega}_t^A := \mathbf{1}^\intercal \boldsymbol{\omega}_t =: \widehat{\omega}_t$ into Agent's equilibrium strategy expression in Proposition 1.2, we immediately have

$$\sum_k \frac{1}{K} \widehat{\omega}_t^2 (r_{k,t+1} - \hat{r}_{k,t+1} + \check{\mu}_{r,k}(\widehat{\omega}_t))^2 \propto (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t))^\intercal (\boldsymbol{r}_{t+1} - \hat{\boldsymbol{r}}_{t+1} + \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t)) =: d(\boldsymbol{r}_{t+1}, \hat{\boldsymbol{r}}_{t+1}),$$

utilizing the fact that for the purpose of extremization, distance metrics are defined only up to a constant of proportionality.

□

D.3 Proof of Theorem 1

Proof. We derive (or simply guess some parts of) the specific solution to the informational problem, then verify that it satisfies the necessary conditions for optimality. Given that objective function is convex, it has a unique minimum, so locating one candidate solution that satisfies the Karush–Kuhn–Tucker conditions suffices.

(a) Interior solution ("large" κ, "small" λ).

The following proof essentially replicates Sims's (2003, 2006) arguments; but this is a classic result in information theory, e.g. see Berger (1971) or Cover and Thomas (2006).

1. $\boldsymbol{x} \in \mathbb{R}^K$. Guess (and verify later) that $\hat{\boldsymbol{x}} \in \mathbb{R}^K$. Hence,

$$\nu(\boldsymbol{x}, \hat{\boldsymbol{x}}) = 0 \quad \forall \boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathbb{R}^K.$$

2. Show that $e^{-\frac{1}{\lambda}\mu(\boldsymbol{x})} = (\pi\lambda)^{-\frac{K}{2}}$ is a valid element of the solution:

$$1 = \int_{\mathbb{R}^K} f(\boldsymbol{x}|\hat{\boldsymbol{x}})\, d\boldsymbol{x} = \int_{\mathbb{R}^K} e^{\frac{1}{\lambda}\nu(\boldsymbol{x},\hat{\boldsymbol{x}}) - \frac{1}{\lambda}\mu(\boldsymbol{x}) - \frac{1}{\lambda}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))^\intercal(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))} =:$$
$$=: \int_{\mathbb{R}^K} b(\boldsymbol{x})(2\pi)^{-\frac{K}{2}} \left|\frac{\lambda}{2}\boldsymbol{I}_K\right|^{-\frac{1}{2}} e^{-\frac{1}{2}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))^\intercal\left(\frac{\lambda}{2}\boldsymbol{I}_K\right)^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))}\, d\boldsymbol{x} =$$
$$= \int_{\mathbb{R}^K} b(\boldsymbol{x})\phi(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t) - \boldsymbol{x} \mid \frac{\lambda}{2}\boldsymbol{I}_K)\, d\boldsymbol{x} =: (b * \phi)\,(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t)).$$

Applying Fourier transform to the convolution above, we get:

$$\delta(\boldsymbol{\xi}) = \widetilde{1} = \widetilde{b}(\boldsymbol{\xi})\widetilde{\phi}(\boldsymbol{\xi}|\frac{\lambda}{2}\boldsymbol{I}_K),$$

$$\widetilde{b}(\boldsymbol{\xi}) = \frac{\delta(\boldsymbol{\xi})}{\widetilde{\phi}(\boldsymbol{\xi}|\frac{\lambda}{2}\boldsymbol{I}_K)} = \delta(\boldsymbol{\xi})e^{2\pi i(\hat{\boldsymbol{x}}-\check{\boldsymbol{\mu}}(\widehat{\omega}_t))\cdot\boldsymbol{\xi}+\pi^2\lambda\boldsymbol{\xi}^\intercal\boldsymbol{\xi}} = \delta(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } \boldsymbol{\xi} = \boldsymbol{0}, \\ 0 & \text{if } \boldsymbol{\xi} \neq \boldsymbol{0}. \end{cases}$$

Inverse Fourier transform gives

$$b(\boldsymbol{x}) = \int_{\mathbb{R}^K} \widetilde{b}(\boldsymbol{\xi})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^K} \delta(\boldsymbol{\xi})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = 1;$$

therefore,

$$e^{-\frac{1}{\lambda}\mu(\boldsymbol{x})} = (2\pi)^{-\frac{K}{2}} \left|\frac{\lambda}{2}\boldsymbol{I}_K\right|^{-\frac{1}{2}} = (\pi\lambda)^{-\frac{K}{2}}.$$

This means

$$\mu(\boldsymbol{x}) = \lambda\frac{K}{2}\ln(\pi\lambda).$$

3. Denoting

$$\boldsymbol{\epsilon} := \boldsymbol{x} - \hat{\boldsymbol{x}} + \check{\boldsymbol{\mu}}(\widehat{\omega}_t) \sim \mathcal{N}(\boldsymbol{0}, \frac{\lambda}{2}\boldsymbol{I}_K),$$

we can represent $\boldsymbol{x}$ as a sum of two different terms, $(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t))$ and "approximation error" $\boldsymbol{\epsilon}$:

$$\boldsymbol{x} = \hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t) + \boldsymbol{\epsilon}.$$

A convolution of independently distributed $(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t))$ with independent $\mathcal{N}$-distributed $\boldsymbol{\epsilon}$ that results in $\mathcal{N}$-distributed $\boldsymbol{x}$ implies $\mathcal{N}$ distribution for $(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t))$ too, and hence also for $\hat{\boldsymbol{x}}$:

$$\hat{\boldsymbol{x}} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}(\widehat{\omega}_t), \hat{\boldsymbol{\Sigma}}).$$

Therefore,

$$\boldsymbol{\Psi} := \mathrm{E}^f[(\boldsymbol{x} - \hat{\boldsymbol{x}} + \check{\boldsymbol{\mu}}(\widehat{\omega}_t))(\boldsymbol{x} - \hat{\boldsymbol{x}} + \check{\boldsymbol{\mu}}(\widehat{\omega}_t))^\intercal] = \mathrm{E}^\phi[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^\intercal] = \frac{\lambda}{2}\boldsymbol{I}_K,$$

and we also have

$$\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}} + \boldsymbol{\Psi}$$

(it can be seen here why we needed uncorrelated random variables: diagonal $\boldsymbol{\Sigma}$ ensures that resulting $\hat{\boldsymbol{\Sigma}}$ is positive semi-definite).

Lastly, conformity of the means (accounting for the bias term) as well as conformity of the variances in case of $\mathcal{N}$ densities $f(\boldsymbol{x}|\hat{\boldsymbol{x}})$ and $h(\hat{\boldsymbol{x}})$ necessarily leads to their product $f(\boldsymbol{x}, \hat{\boldsymbol{x}})$ satisfying the constraint $\int_{\mathrm{supp}(h)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\hat{\boldsymbol{x}} = g(\boldsymbol{x})\ \forall \boldsymbol{x} \in \mathrm{supp}(g)$ (the $\mu(\boldsymbol{x})$-constraint).

4. Information processing capacity constraint reduces to

$$\kappa = \frac{1}{2}(\ln|\boldsymbol{\Sigma}| - \ln|\boldsymbol{\Psi}|) = \frac{1}{2}\left(\ln|\boldsymbol{\Sigma}| - \ln\left|\frac{\lambda}{2}\boldsymbol{I}_K\right|\right),$$

which means that

$$\lambda = 2\left(e^{-2\kappa}|\boldsymbol{\Sigma}|\right)^{\frac{1}{K}}.$$

(b) Boundary solution ("small" κ, "large" λ).

The essense of this proof is known in information theory as "reverse water-filling" solution, or more accurately "reverse water-filling on the eigenvalues" for cases like ours, e.g. see Berger (1971). This procedure is more general than what was used for interior solution.

0. If the condition $\sigma_k^2 > \frac{\lambda}{2} \quad \forall k \in \{1, \ldots, K\}$ doesn't hold, i.e. $\exists k \in \{1, \ldots, K\} : \sigma_k^2 \leq \frac{\lambda}{2}$, the interior solution violates the constraint $\int_{\mathrm{supp}(h)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\hat{\boldsymbol{x}} = g(\boldsymbol{x})\ \forall \boldsymbol{x} \in \mathrm{supp}(g)$ (the $\mu(\boldsymbol{x})$-constraint).

1. $\boldsymbol{x} \in \mathbb{R}^K$. Look for solution with $\hat{\boldsymbol{x}} \in \mathbb{R}^{k^*} \times \{\hat{\mu}_{k^*+1}(\widehat{\omega}_t), \cdots, \hat{\mu}_K(\widehat{\omega}_t)\}$ (otherwise, $\epsilon_{k^*+1}, \cdots, \epsilon_K$ will have non-zero means in order to satisfy the $\mu(\boldsymbol{x})$-constraint). Hence,

$$\nu(\boldsymbol{x}, \hat{\boldsymbol{x}}) = 0 \quad \forall \boldsymbol{x} \in \mathbb{R}^K, \forall \hat{\boldsymbol{x}} \in \mathbb{R}^{k^*} \times \{\hat{\mu}_{k^*+1}(\widehat{\omega}_t), \cdots, \hat{\mu}_K(\widehat{\omega}_t)\}.$$

2. Show that $e^{-\frac{1}{\lambda}\mu(\boldsymbol{x})} = (2\pi)^{-\frac{K}{2}} \left(\left(\frac{\lambda}{2}\right)^{k^*} \sigma_{k^*+1}^2 \times \cdots \times \sigma_K^2\right)^{-\frac{1}{2}}$ is a valid element of the solution:

$$\begin{aligned} 1 &= \int_{\mathbb{R}^K} f(\boldsymbol{x}|\hat{\boldsymbol{x}})\, d\boldsymbol{x} = \int_{\mathbb{R}^K} e^{\frac{1}{\lambda}\nu(\boldsymbol{x},\hat{\boldsymbol{x}})-\frac{1}{\lambda}\mu(\boldsymbol{x})-\frac{1}{\lambda}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))^\intercal(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))}\, d\boldsymbol{x} =: \\ &=: \int_{\mathbb{R}^K} b(\boldsymbol{x})(2\pi)^{-\frac{K}{2}}\, |\boldsymbol{\Psi}|^{-\frac{1}{2}}\, e^{-\frac{1}{2}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))^\intercal \boldsymbol{\Psi}^{-1}(\boldsymbol{x}-\hat{\boldsymbol{x}}+\check{\boldsymbol{\mu}}(\widehat{\omega}_t))}\, d\boldsymbol{x} = \\ &= \int_{\mathbb{R}^K} b(\boldsymbol{x})\phi(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t) - \boldsymbol{x} \mid \boldsymbol{\Psi})\, d\boldsymbol{x} =: (b * \phi)\,(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t)). \end{aligned}$$

Applying Fourier transform to the convolution above, we get:

$$\delta(\boldsymbol{\xi}) = \widetilde{1} = \widetilde{b}(\boldsymbol{\xi})\widetilde{\phi}(\boldsymbol{\xi}|\boldsymbol{\Psi}),$$

$$\begin{aligned} \widetilde{b}(\boldsymbol{\xi}) &= \frac{\delta(\boldsymbol{\xi})}{\widetilde{\phi}(\boldsymbol{\xi}|\boldsymbol{\Psi})} = \delta(\boldsymbol{\xi}) e^{2\pi i(\hat{\boldsymbol{x}}-\check{\boldsymbol{\mu}}(\widehat{\omega}_t))\cdot\boldsymbol{\xi}+\pi^2(\lambda\sum_1^{k^*}\xi_k^2+2\sum_{k^*+1}^K \sigma_k^2\xi_k^2)} = \\ &= \delta(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } \boldsymbol{\xi} = \boldsymbol{0}, \\ 0 & \text{if } \boldsymbol{\xi} \neq \boldsymbol{0}. \end{cases} \end{aligned}$$

Inverse Fourier transform gives

$$b(\boldsymbol{x}) = \int_{\mathbb{R}^K} \widetilde{b}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^K} \delta(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = 1,$$

therefore,

$$e^{-\frac{1}{\lambda}\mu(\boldsymbol{x})} = (2\pi)^{-\frac{K}{2}}\, |\boldsymbol{\Psi}|^{-\frac{1}{2}} = (2\pi)^{-\frac{K}{2}} \left(\left(\frac{\lambda}{2}\right)^{k^*} \sigma_{k^*+1}^2 \times \cdots \times \sigma_K^2\right)^{-\frac{1}{2}}.$$

This means

$$\mu(\boldsymbol{x}) = \lambda\left(\frac{K}{2}\ln(2\pi) + \frac{1}{2}\ln\left(\left(\frac{\lambda}{2}\right)^{k^*} \sigma_{k^*+1}^2 \times \cdots \times \sigma_K^2\right)\right).$$

3. Denoting

$$\boldsymbol{\epsilon} := \boldsymbol{x} - \hat{\boldsymbol{x}} + \check{\boldsymbol{\mu}}(\widehat{\omega}_t) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Psi}),$$

we can represent $\boldsymbol{x}$ as a sum of two different terms, $(\hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t))$ and "approximation error" $\boldsymbol{\epsilon}$:

$$\boldsymbol{x} = \hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t) + \boldsymbol{\epsilon}.$$

A convolution of independently distributed $(\hat{\boldsymbol{x}}-\check{\boldsymbol{\mu}}(\widehat{\omega}_t))$ with independent $\mathcal{N}$-distributed $\boldsymbol{\epsilon}$ that results in $\mathcal{N}$-distributed $\boldsymbol{x}$ implies $\mathcal{N}$ distribution for $(\hat{\boldsymbol{x}}-\check{\boldsymbol{\mu}}(\widehat{\omega}_t))$ too, and hence also for $\hat{\boldsymbol{x}}$:

$$\hat{\boldsymbol{x}} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}(\widehat{\omega}_t), \hat{\boldsymbol{\Sigma}}).$$

Therefore (with $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_K\}$ being the standard basis of $\mathbb{R}^K$),

$$\boldsymbol{\Psi} := \mathrm{E}^f[(\boldsymbol{x} - \hat{\boldsymbol{x}} + \check{\boldsymbol{\mu}}(\widehat{\omega}_t))(\boldsymbol{x} - \hat{\boldsymbol{x}} + \check{\boldsymbol{\mu}}(\widehat{\omega}_t))^\intercal] = \mathrm{E}^\phi[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^\intercal] = \begin{bmatrix} \frac{\lambda}{2}\boldsymbol{I}_{k^*} & \boldsymbol{0} \\ \boldsymbol{0} & \sum_{k=1}^{K-k^*} \boldsymbol{e}_k^\intercal \boldsymbol{\Sigma}_{k^*+1K} \boldsymbol{e}_k \end{bmatrix},$$

and we also have

$$\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}} + \boldsymbol{\Psi}$$

(it can be seen here why we needed uncorrelated random variables: diagonal $\boldsymbol{\Sigma}$ ensures that resulting $\hat{\boldsymbol{\Sigma}}$ is positive semi-definite).

Lastly, conformity of the means (accounting for the bias term) as well as conformity of the variances in case of $\mathcal{N}$ densities $f(\boldsymbol{x}|\hat{\boldsymbol{x}})$ and $h(\hat{\boldsymbol{x}})$ necessarily leads to their product $f(\boldsymbol{x}, \hat{\boldsymbol{x}})$ satisfying the constraint $\int_{\mathrm{supp}(h)} f(\boldsymbol{x}, \hat{\boldsymbol{x}})\, d\hat{\boldsymbol{x}} = g(\boldsymbol{x})\ \forall \boldsymbol{x} \in \mathrm{supp}(g)$ (the $\mu(\boldsymbol{x})$-constraint).

4. Information processing capacity constraint reduces to

$$\kappa = \frac{1}{2}(\ln|\boldsymbol{\Sigma}| - \ln|\boldsymbol{\Psi}|) = \frac{1}{2}\left(\ln|\boldsymbol{\Sigma}| - \ln\left|\frac{\lambda}{2}\boldsymbol{I}_{k^*}\right| - \ln\left|\sum_{k=1}^{K-k^*} \boldsymbol{e}_k^\intercal \boldsymbol{\Sigma}_{k^*+1K} \boldsymbol{e}_k\right|\right),$$

which means that

$$\lambda = 2\left(e^{-2\kappa}\sigma_{k^*+1}^{-2} \cdots \sigma_K^{-2}|\boldsymbol{\Sigma}|\right)^{\frac{1}{k^*}}.$$

□

D.4 Proof of Theorem 2

Proof. Premultiplying with $\boldsymbol{\Xi}$ equation $\boldsymbol{x} = \hat{\boldsymbol{x}} - \check{\boldsymbol{\mu}}(\widehat{\omega}_t) + \boldsymbol{\epsilon}$ produces

$$\boldsymbol{r}_{t+1} = \hat{\boldsymbol{r}}_{t+1} - \check{\boldsymbol{\mu}}_r(\widehat{\omega}_t) + \boldsymbol{\epsilon}_{r,t+1},$$

using (I.53), (I.54), (I.55), and defining $\boldsymbol{\epsilon}_{r,t+1} := \boldsymbol{\Xi}\boldsymbol{\epsilon}$.

Premultiplying with $\boldsymbol{\Xi}$ and postmultiplying with $\boldsymbol{\Xi}^{-1}$ equation $\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}} + \boldsymbol{\Psi}$ produces

$$\boldsymbol{\Sigma}_r = \hat{\boldsymbol{\Sigma}}_r + \boldsymbol{\Psi}_r,$$

as from formulas in $\mathcal{P}_{\boxslash}$, $\hat{\boldsymbol{\Sigma}}_r := \boldsymbol{\Xi}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Xi}^{-1}$ and $\boldsymbol{\Psi}_r := \boldsymbol{\Xi}\boldsymbol{\Psi}\boldsymbol{\Xi}^{-1}$.

Given that $\hat{\boldsymbol{x}} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}(\widehat{\omega}_t), \hat{\boldsymbol{\Sigma}})$, $\hat{\boldsymbol{r}}_{t+1} = \boldsymbol{\Xi}\hat{\boldsymbol{x}}$ (from equation I.54) is distributed as $\mathcal{N}(\boldsymbol{\Xi}\hat{\boldsymbol{\mu}}(\widehat{\omega}_t), \boldsymbol{\Xi}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Xi}^\intercal)$. Since before the decorrelating transformation mean of $\hat{\boldsymbol{r}}_{t+1}$ was $\hat{\boldsymbol{\mu}}_r(\widehat{\omega}_t)$ by Proposition 1.1, and also using relationships in $\mathcal{P}_{\boxslash}$, we have $\hat{\boldsymbol{r}}_{t+1} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}_r(\widehat{\omega}_t), \hat{\boldsymbol{\Sigma}}_r)$.

Given that $\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Psi})$, $\boldsymbol{\epsilon}_{r,t+1} = \boldsymbol{\Xi}\boldsymbol{\epsilon}$ (by the definition from above) is distributed as $\mathcal{N}(\boldsymbol{\Xi}\boldsymbol{0}, \boldsymbol{\Xi}\boldsymbol{\Psi}\boldsymbol{\Xi}^\intercal)$. Using relationships in $\mathcal{P}_{\boxslash}$ we get $\boldsymbol{\epsilon}_{r,t+1} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Psi}_r)$.

□

D.5 Proof of Proposition B.1

Proof. Let $\boldsymbol{\theta}_{\iota-1} := \{q_{0,t,\iota-1}, \boldsymbol{q}_{t,\iota-1}\}$ be the proposed parameter choice that has been accepted at iteration $(\iota - 1)$. By construction, it approaches the optimal parameter choice $\boldsymbol{\theta}^*$, i.e. $||\boldsymbol{\theta}_{\iota-1} - \boldsymbol{\theta}^*||_2^2 \to 0$ as ι increases. The corresponding update of the bias of the mean can then be represented by

$$\begin{aligned}
\left|\left|\check{\boldsymbol{\mu}}_{r,\iota-1} - \check{\boldsymbol{\mu}}_r\right|\right|_2^2 &= \left|\left|\frac{1}{2}\text{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_{r,\iota-1}) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_{r,\iota-1})\boldsymbol{\omega}_{t,\iota-1} - \frac{1}{2}\text{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r) + \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_r)\boldsymbol{\omega}_t\right|\right. \\
&= \left|\left|\frac{1}{2}\text{diag}^{-1}(\boldsymbol{\Psi}_{r,\iota-1}) - \frac{1}{2}\boldsymbol{\Psi}_{r,\iota-1}\boldsymbol{\omega}_{t,\iota-1} - \frac{1}{2}\text{diag}^{-1}(\boldsymbol{\Psi}_r) + \frac{1}{2}\boldsymbol{\Psi}_r\boldsymbol{\omega}_t\right|\right|_2^2 = \\
&= \left|\left|\frac{1}{2}\text{diag}^{-1}(\boldsymbol{\Psi}_r) - \frac{1}{2}\boldsymbol{\Psi}_r\boldsymbol{\omega}_{t,\iota-1} - \frac{1}{2}\text{diag}^{-1}(\boldsymbol{\Psi}_r) + \frac{1}{2}\boldsymbol{\Psi}_r\boldsymbol{\omega}_t\right|\right|_2^2 = \\
&= \frac{1}{2}\left|\left|\boldsymbol{\Psi}_r\left(\boldsymbol{\omega}_{t,\iota-1} - \boldsymbol{\omega}_t\right)\right|\right|_2^2 = \\
&= \frac{1}{2}\left|\left|\boldsymbol{\Psi}_r\left(\frac{1}{W_t}\text{diag}(\boldsymbol{P}_t)\boldsymbol{q}_{t,\iota-1} - \frac{1}{W_t}\text{diag}(\boldsymbol{P}_t)\boldsymbol{q}_t^*\right)\right|\right|_2^2 = \\
&= \frac{1}{2}\left|\left|\boldsymbol{\Psi}_r\frac{1}{W_t}\text{diag}(\boldsymbol{P}_t)\left(\boldsymbol{q}_{t,\iota-1} - \boldsymbol{q}_t^*\right)\right|\right|_2^2 \leq \\
&\leq \frac{1}{2}\left|\left|\boldsymbol{\Psi}_r\frac{1}{W_t}\text{diag}(\boldsymbol{P}_t)\right|\right|_2^2\left|\left|\boldsymbol{q}_{t,\iota-1} - \boldsymbol{q}_t^*\right|\right|_2^2,
\end{aligned}$$

where the third equality is due to invariance of the solution to informational problem (modulo specific values of the bias term), and the weak inequality is due to consistency of induced matrix norm. Clearly, improvement in $\boldsymbol{q}_{t,\iota-1}$ directly leads to improvement in $\check{\boldsymbol{\mu}}_{r,\iota-1}$.

We are dealing with a well-posed consumption-investment maximization problem that posesses a unique maximum to which the above iterative optimization procedure converges. Covergence at some iteration ι by definition means that (making the dependence of $\varphi(\cdot|\boldsymbol{\theta})$ on $\hat{\boldsymbol{\mu}}_r$ and in turn on $\check{\boldsymbol{\mu}}_r$ explicit)

$$\left|\left|\varphi(\boldsymbol{\Xi}^\intercal\hat{\boldsymbol{r}}_{t+1}|\boldsymbol{\theta}_\iota, \hat{\boldsymbol{\mu}}_r(\check{\boldsymbol{\mu}}_{r,\iota-1})) - \varphi(\boldsymbol{\Xi}^\intercal\hat{\boldsymbol{r}}_{t+1}|\boldsymbol{\theta}^*, \hat{\boldsymbol{\mu}}_r(\check{\boldsymbol{\mu}}_r))\right|\right|_2^2 < \varepsilon_\varphi$$

for some pre-specified $\varepsilon_\varphi > 0$. This corresponds to covergence in chosen parameters:

$$\left|\left|\boldsymbol{q}_{t,\iota} - \boldsymbol{q}_t^*\right|\right|_2^2 \leq ||\boldsymbol{\theta}_\iota - \boldsymbol{\theta}^*||_2^2 < \varepsilon_\theta$$

for some small $\varepsilon_\theta > 0$, as well as in bias of the mean:

$$||\check{\boldsymbol{\mu}}_{r,\iota} - \check{\boldsymbol{\mu}}_r||_2^2 \leq \frac{1}{2}\left|\left|\boldsymbol{\Psi}_r\frac{1}{W_t}\text{diag}(\boldsymbol{P}_t)\right|\right|_2^2\left|\left|\boldsymbol{q}_{t,\iota} - \boldsymbol{q}_t^*\right|\right|_2^2 < M\varepsilon_\theta =: \varepsilon_\mu$$

for some $0 < M < \infty$ (finiteness of M is guaranteed by stationarity of the environment).

According to optimization procedure considered here, upon reaching the optimum we have

$$\boldsymbol{q}_{t,\iota} = \boldsymbol{q}_{t,\iota-1},$$

with a related implication for updating of the bias term:

$$\check{\boldsymbol{\mu}}_{r,\iota} = \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_{r,\iota}) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_{r,\iota})\boldsymbol{\omega}_{t,\iota} = \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Psi}_r) - \frac{1}{2}\boldsymbol{\Psi}_r\frac{1}{W_t}\mathrm{diag}(\boldsymbol{P}_t)\boldsymbol{q}_{t,\iota} =$$
$$= \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Psi}_r) - \frac{1}{2}\boldsymbol{\Psi}_r\frac{1}{W_t}\mathrm{diag}(\boldsymbol{P}_t)\boldsymbol{q}_{t,\iota-1} = \frac{1}{2}\mathrm{diag}^{-1}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_{r,\iota-1}) - \frac{1}{2}(\boldsymbol{\Sigma}_r - \hat{\boldsymbol{\Sigma}}_{r,\iota-1})\boldsymbol{\omega}_{t,\iota-1} = \check{\boldsymbol{\mu}}_{r,\iota-1}.$$

Which is immediately reftected in update of the mean, verifying the convergence:

$$||\varphi(\boldsymbol{x}|\boldsymbol{\theta}_{\iota-1}) - \varphi(\boldsymbol{x}|\boldsymbol{\theta}^*)||_2^2 = \left|\left|\varphi(\boldsymbol{\Xi}^\intercal\hat{\boldsymbol{r}}_{t+1}|\boldsymbol{\theta}_\iota, \hat{\boldsymbol{\mu}}_r(\check{\boldsymbol{\mu}}_{r,\iota})) - \varphi(\boldsymbol{\Xi}^\intercal\hat{\boldsymbol{r}}_{t+1}|\boldsymbol{\theta}^*, \hat{\boldsymbol{\mu}}_r(\check{\boldsymbol{\mu}}_r))\right|\right|_2^2 =$$
$$= \left|\left|\varphi(\boldsymbol{\Xi}^\intercal\hat{\boldsymbol{r}}_{t+1}|\boldsymbol{\theta}_\iota, \hat{\boldsymbol{\mu}}_r(\check{\boldsymbol{\mu}}_{r,\iota-1})) - \varphi(\boldsymbol{\Xi}^\intercal\hat{\boldsymbol{r}}_{t+1}|\boldsymbol{\theta}^*, \hat{\boldsymbol{\mu}}_r(\check{\boldsymbol{\mu}}_r))\right|\right|_2^2 < \varepsilon_\varphi.$$

Finalizing the argument, at every iteration Theorems 1–2 hold with bias term $\check{\boldsymbol{\mu}}_{r,\iota}$ and mean $\hat{\boldsymbol{\mu}}_{r,\iota}$. Moreover, at the optimum $\check{\boldsymbol{\mu}}_{r,\iota}$, $\hat{\boldsymbol{\mu}}_{r,\iota}$ and $\boldsymbol{\omega}_{t,\iota}$ are consistent with definitions for $\check{\boldsymbol{\mu}}_r$, $\hat{\boldsymbol{\mu}}_r$ and $\boldsymbol{\omega}_t$ in Proposition 1.1. This gives the stated result.

□

D.6 Proof of Corollary 2

Proof. For $\boldsymbol{r}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}_r, \boldsymbol{\Sigma}_r)$ (equation 8), with $\boldsymbol{\Sigma}_r = \boldsymbol{\Xi}\boldsymbol{\Sigma}\boldsymbol{\Xi}^{-1}$ and $\boldsymbol{\Sigma}$ diagonal (using formulas in $\mathcal{P}_{\boxslash}$), we have

$$\rho_{r,kl} := \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}\sigma_m^2}{\left(\sum_{m=1}^K \xi_{km}^2\sigma_m^2\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2\sigma_m^2\right)^{1/2}}.$$

From Theorem 2 we obtain $\hat{\boldsymbol{r}}_{t+1} \sim \mathcal{N}(\hat{\boldsymbol{\mu}}_r, \hat{\boldsymbol{\Sigma}}_r)$, where $\hat{\boldsymbol{\Sigma}}_r = \boldsymbol{\Xi}\hat{\boldsymbol{\Sigma}}\boldsymbol{\Xi}^{-1}$ with diagonal $\hat{\boldsymbol{\Sigma}}$, hence

$$\hat{\rho}_{r,kl} := \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}\hat{\sigma}_m^2}{\left(\sum_{m=1}^K \xi_{km}^2\hat{\sigma}_m^2\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2\hat{\sigma}_m^2\right)^{1/2}} =$$
$$= \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}(\sigma_m^2 - \psi_m^2)}{\left(\sum_{m=1}^K \xi_{km}^2(\sigma_m^2 - \psi_m^2)\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2(\sigma_m^2 - \psi_m^2)\right)^{1/2}}.$$

In the interior solution case, $\psi_m^2 = \psi_n^2, \forall m, n \in \{1, \ldots, K\}$, thus

$$\hat{\rho}_{r,kl} = \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}(\sigma_m^2 - \psi_1^2)}{\left(\sum_{m=1}^K \xi_{km}^2(\sigma_m^2 - \psi_1^2)\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2(\sigma_m^2 - \psi_1^2)\right)^{1/2}} =$$
$$= \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}\sigma_m^2}{\left(\sum_{m=1}^K \xi_{km}^2\sigma_m^2 - \psi_1^2\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2\sigma_m^2 - \psi_1^2\right)^{1/2}},$$

with the last equality following from orthogonality of the matrix of eigenvectors $\boldsymbol{\Xi}$, i.e. from the fact that $\sum_{m=1}^K \xi_{km}\xi_{lm} = \delta_{kl}$, where δ_{kl} is Kronecker delta function returning 1 when $k = l$ and 0 otherwise. Simple algebraic manipulations deliver the stated relationship:

$$\hat{\rho}_{r,kl} = \rho_{r,kl} \times \frac{\left(\sum_{m=1}^K \xi_{km}^2\sigma_m^2\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2\sigma_m^2\right)^{1/2}}{\left(\sum_{m=1}^K \xi_{km}^2\sigma_m^2 - \psi_1^2\right)^{1/2}\left(\sum_{m=1}^K \xi_{lm}^2\sigma_m^2 - \psi_1^2\right)^{1/2}}, \qquad \forall k, l \in \{1, \ldots, K\},$$

with the fraction term on the right clearly being larger than or equal to 1, thus pushing $|\hat{\rho}_{r,kl}|$ from $|\rho_{r,kl}|$ towards 1. Lastly, Theorem 1 provides the value of ψ_1^2.

In the boundary solution case, $\psi_m^2 \neq \psi_n^2, \forall m, n \in \{1, \dots, K\}$ in general, so a simple argument from above does not go through. The proof is based instead on providing two contrasting examples. Given set $\{\sigma_k^2\}_1^K$ sorted in descending order, for each $k \in \{1, \dots, K\}$ take

$$\psi_k^2 := \begin{cases} \sigma_K^2 & \text{if } k = K, \\ \sigma_K^2 + \psi_+^2 & \text{if } k \neq K, \end{cases}$$

where $\sigma_k^2 - \sigma_K^2 > \psi_+^2 \geq 0$ for every $k \in \{1, \dots, K-1\}$. Then we have:

$$\begin{aligned} \hat{\rho}_{r,kl} =& \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}(\sigma_m^2 - \sigma_K^2) - \sum_{m=1}^{K-1} \xi_{km}\xi_{lm}\psi_+^2}{\left(\sum_{m=1}^K \xi_{km}^2(\sigma_m^2 - \sigma_K^2) - \sum_{m=1}^{K-1} \xi_{km}^2\psi_+^2\right)^{1/2} \left(\sum_{m=1}^K \xi_{lm}^2(\sigma_m^2 - \sigma_K^2) - \sum_{m=1}^{K-1} \xi_{lm}^2\psi_+^2\right)^{1/2}} = \\ =& \frac{\sum_{m=1}^K \xi_{km}\xi_{lm}\sigma_m^2 + \xi_{kK}\xi_{lK}\psi_+^2}{\left(\sum_{m=1}^K \xi_{km}^2\sigma_m^2 - \sigma_K^2 - \psi_+^2 + \xi_{kK}^2\psi_+^2\right)^{1/2} \left(\sum_{m=1}^K \xi_{lm}^2\sigma_m^2 - \sigma_K^2 - \psi_+^2 + \xi_{lK}^2\psi_+^2\right)^{1/2}}, \end{aligned}$$

with the last step using orthogonality of the matrix of eigenvectors $\boldsymbol{\Xi}$. Now consider the limit of $\hat{\rho}_{r,kl}$ when σ_K^2 becomes negligibly small:

$$\hat{\rho}_{r,kl,\text{-}K} := \lim_{\sigma_K^2 \to 0+} \hat{\rho}_{r,kl} = \frac{\sum_{m=1}^{K-1} \xi_{km}\xi_{lm}\sigma_m^2 + \xi_{kK}\xi_{lK}\psi_+^2}{\left(\sum_{m=1}^{K-1} \xi_{km}^2\sigma_m^2 - \psi_+^2 + \xi_{kK}^2\psi_+^2\right)^{1/2} \left(\sum_{m=1}^{K-1} \xi_{lm}^2\sigma_m^2 - \psi_+^2 + \xi_{lK}^2\psi_+^2\right)^{1/2}}.$$

Evaluated at $\psi_+^2 := 0$, this limit for $\hat{\rho}_{r,kl}$ would coincide with its counterpart for $\rho_{r,kl}$ irrespective of the values of ξ_{kK} and ξ_{lK}:

$$\begin{aligned} \hat{\rho}_{r,kl,\text{-}K}\Big|_{\psi_+^2=0} &= \lim_{\sigma_K^2 \to 0+} \hat{\rho}_{r,kl}\Big|_{\psi_+^2=0} = \frac{\sum_{m=1}^{K-1} \xi_{km}\xi_{lm}\sigma_m^2}{\left(\sum_{m=1}^{K-1} \xi_{km}^2\sigma_m^2\right)^{1/2} \left(\sum_{m=1}^{K-1} \xi_{lm}^2\sigma_m^2\right)^{1/2}} = \\ &= \lim_{\sigma_K^2 \to 0+} \rho_{r,kl}\Big|_{\psi_+^2=0} = \rho_{r,kl,\text{-}K}\Big|_{\psi_+^2=0}. \end{aligned}$$

Assume $\rho_{r,kl,\text{-}K} > 0$. Choose $\xi_{kK} = \pm 1/\sqrt{2}$ and $\xi_{lK} = \mp 1/\sqrt{2}$, which produces

$$\hat{\rho}_{r,kl,\text{-}K} = \frac{\sum_{m=1}^{K-1} \xi_{km}\xi_{lm}\sigma_m^2 - \frac{1}{2}\psi_+^2}{\left(\sum_{m=1}^{K-1} \xi_{km}^2\sigma_m^2 - \frac{1}{2}\psi_+^2\right)^{1/2} \left(\sum_{m=1}^{K-1} \xi_{lm}^2\sigma_m^2 - \frac{1}{2}\psi_+^2\right)^{1/2}}.$$

Given that expression $\hat{\rho}_{r,kl,\text{-}K}$ is continuous and differentiable on the whole interval of admissible ψ_+^2, we can take respective derivative at point $\psi_+^2 = 0$, obtaining (after some rearrangement):

$$\left.\frac{\partial\hat{\rho}_{r,kl,\text{-}K}}{\partial\psi_+^2}\right|_{\psi_+^2=0} = \hat{\rho}_{r,kl,\text{-}K} \times \left(\frac{1}{2}\frac{1}{\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2 - \frac{1}{2}\psi_+^2} + \frac{1}{2}\frac{1}{\sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2 - \frac{1}{2}\psi_+^2}\right)\Bigg|_{\psi_+^2=0} -$$

$$- \left.\frac{1}{\left(\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2 - \frac{1}{2}\psi_+^2\right)^{1/2}\left(\sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2 - \frac{1}{2}\psi_+^2\right)^{1/2}}\right|_{\psi_+^2=0} =$$

$$= \rho_{r,kl,\text{-}K} \times \left(\frac{1}{2}\frac{1}{\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2} + \frac{1}{2}\frac{1}{\sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2}\right) -$$

$$- \frac{1}{\left(\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2\right)^{1/2}\left(\sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2\right)^{1/2}}.$$

By Young's inequality for products,

$$\frac{1}{2}\frac{1}{\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2} + \frac{1}{2}\frac{1}{\sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2} \geq \frac{1}{\left(\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2\right)^{1/2}\left(\sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2\right)^{1/2}},$$

with equality holding if and only if $\sum_{m=1}^{K-1}\xi_{km}^2\sigma_m^2 = \sum_{m=1}^{K-1}\xi_{lm}^2\sigma_m^2$. Unless the equality does hold, there always exists $\rho_{r,kl,\text{-}K} \in (0,1]$ such that $\partial\hat{\rho}_{r,kl,\text{-}K}/\partial\psi_+^2\big|_{\psi_+^2=0} > 0$, in which case $\hat{\rho}_{r,kl}$ moves away from $\rho_{r,kl}$ towards 1; and even without invocation of Young's inequality, it's trivial to find $\rho_{r,kl,\text{-}K} \in (0,1]$ such that $\partial\hat{\rho}_{r,kl,\text{-}K}/\partial\psi_+^2\big|_{\psi_+^2=0} < 0$, in which case $\hat{\rho}_{r,kl}$ moves away from $\rho_{r,kl}$ towards 0. If $\rho_{r,kl,\text{-}K} = 0$, we immediately get $\partial\hat{\rho}_{r,kl,\text{-}K}/\partial\psi_+^2\big|_{\psi_+^2=0} < 0$, in which case $\hat{\rho}_{r,kl}$ moves away from $\rho_{r,kl} = 0$ towards -1. (The situation under assumption of $\rho_{r,kl,\text{-}K} < 0$ can be handled similarly.)

□